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Mathematical Reviews

Vol. 13, No. 10 NOVEMBER, 1952 Pages 897–1014

FOUNDATIONS

Mostowski, Andrzej. On direct products of theories. J. Symbolic Logic 17, 1–31 (1952).

F sei eine Menge und $R_1, \cdots, R_s$ seien Relationen in F. Die Aussageformen, die aus $R_1, \cdots, R_s$ mit den Konstanten der elementaren Prädikatenlogik zusammengesetzt sind, heissen "elementar" bzgl. $R_1, \cdots, R_s$. Die Menge der elementaren Sätze bzgl. $R_1, \cdots, R_s$ heisst die "elementare Theorie" T von $R_1, \cdots, R_s$. Für eine Menge I sei F^I die Menge der Abbildungen f von I in F. Für jede Relation R in F sei $R'(f_1, \cdots, f_n)$ definiert durch: $R(f_1(a), \cdots, f_n(a))$ für alle $a \epsilon I$. Die elementare Theorie T^I von $R_1', \cdots, R_s'$ heisst eine starke Potenz von T. Für unendliches I und ein $e \epsilon F^I$ sei ${}^*F_e{}^I$ die Menge der f mit $f(a) = e(a)$ für fast alle a. Es sei $R'' = R' \cap ({}^*F_e{}^I)^n$. Die elementare Theorie von $R_1'', \cdots, R_s''$ heisst eine schwache Potenz ${}^*T_e{}^I$ von T. Das wichtige Resultat dieser Arbeit ist, dass T^I und ${}^*T_e{}^I$ (für konstantes e) entscheidbar sind, wenn T entscheidbar ist. Der Beweis baut auf der Entscheidbarkeit der elementaren Theorie der Inklusion $\subset$ in der Potenzmenge I_1 von I [Th. Skolem, Skr. Vid. Kristiania. I. **1919**, no. 3] auf. Durch sukzessive Elimination von Quantoren in der elementaren Theorie von $I_1, \subset, F, R_1, \cdots, R_s, F^I, \epsilon, Q$ mit $Q(f, a, x) \equiv f(a) = x$ lässt sich die Entscheidbarkeit von T^I auf die Entscheidbarkeit von T zurückführen. Für schwache Potenzen läuft der Beweis weitgehend parallel. Verf. bemerkt zum Schluss, dass sich die Entscheidbarkeit im allgemeinen nicht auf direkte Produkte verschiedener Theorien überträgt.

P. Lorenzen (Bonn).

Quine, W. V. On an application of Tarski's theory of truth. Proc. Nat. Acad. Sci. U. S. A. 38, 430–433 (1952).

If L is the system of the author's "Mathematical logic" [rev. ed., Harvard Univ. Press, 1951; these Rev. 13, 613] and L' is L plus the protosyntax of L, then by a result of Tarski, if L is consistent, truth in L cannot be definable in L'. If x denotes a function which assigns an entity to each variable, and y a formula in L, Tarski's recursive definition of "x satisfies y" must be somewhat modified in order to account for the occurrence of non-elements in the interpretation of L; this recursive definition of satisfaction cannot be turned into a direct definition, unless L is inconsistent.

A. Heyting (Amsterdam).

Suppes, Patrick. A set of independent axioms for extensive quantities. Portugaliae Math. 10, 163–172 (1951).

The author points out two defects in previous work on the subject (of which he considers, in particular, O. Hölder [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 53, 1–64 (1901)]). He then characterizes a system of extensive quantities as an ordered triple $\langle K, Q, *\rangle$ of a non-empty set K and a binary relation Q and a binary function $*$, defined over K, which satisfies the following axioms ($x, y, z, \cdots$ are elements of K, $m, n, \cdots$ are natural numbers, nx is defined by recursion): (A I) If xQy and yQz, then xQz; (A II) $x*y$ is in K; (A III) $(x*y)*zQx*(y*z)$; (A IV) if xQy, then $x*zQz*y$; (A V) if not xQy, then there is a z such that $xQy*z$ and $y*zQx$; (A VI) not $x+yQx$; (A VII) if xQy, then there is a number n such that $yQnx$. A proof is given both of the independence of the axioms and of the independence of the primitive notions. Now the relation C, defined by $xCy =_{df} (xQy$ and $yQx)$, is an equivalence relation. So a definition by abstraction enables us to replace the system of extensive quantities $\mathfrak{M} = \langle K, Q, *\rangle$ by a system $\mathfrak{M}/C = \langle K/C, \leqq, +\rangle$ which is called a system of extensive magnitudes. In two metatheorems, it is pointed out that $\mathfrak{M}/C$ is isomorphic to an additive semi-group of positive real numbers, closed under subtraction of smaller numbers from larger ones, and that any two additive semi-groups isomorphic to $\mathfrak{M}/C$ are related by a similarity transformation. This implies the formal adequacy of the axioms.

E. W. Beth (Amsterdam).

Crespo Pereira, Ramón. On Schröder's algebra of logic. Revista Mat. Hisp.-Amer. (4) 11, 222–239 (1951). (Spanish)

The author presents a brief exposition of the first volume of E. Schröder's Vorlesungen über die Algebra der Logik [Teubner, Leipzig, 1890]. In the attempt to follow Schröder too closely he does not postulate the existence of sum, product, or negative; and consequently we have the absurd result that the distributive law is a consequence of the axioms for partial order, the existence of 0 and 1, and the axiom

$$bc = 0 \rightarrow a(b+c) \leqq ab + ac.$$

(Counter examples are the circles in the plane, or the lattice formed by interpolating a 5-element non-distributive lattice between a and b in the four element chain $0 \leqq a \leqq b \leqq 1$.)

H. B. Curry (State College, Pa.).

Dienes, Paul. On H-matrices. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 32–36 (1952).

The author defines a regular logical matrix as an algebra of four operations $+$, $\cdot$, $^-$, and $\rightarrow$ on a set of acceptance and rejection values which satisfies the following three conditions: (1) There is only one acceptance value (call it "1"). (2) $x \rightarrow x$ has the value 1. (3) If both $x \rightarrow y$ and $y \rightarrow x$ have value 1 then x and y have the same value. Let 0 be the value corresponding to any identically false propositional function of the classical calculus. (A theorem demonstrates an isomorphism of equivalence of formulas of a system to equality of values in a regular matrix model.) Let J be the set-theoretic difference of the theorems of classical and intuitionistic propositional calculi. Values taken by the corresponding functions in a matrix are called J-values. It is shown that if a logical matrix H is a model of the intuitionistic propositional calculus it must satisfy the following conditions. (i) If $\bar{x} = 1$, then $x = 0$. (ii) There are J-values distinct from 0 and 1. (iii) Each of the following rules is impossible for H: (a) If $x \neq 0$, then $\bar{x} = 0$; (b) $x + y = \max(x, y)$; (c) $x \cdot y = \min(x, y)$; (d) $x \rightarrow y$ is 1 when $x \leqq y$, and y when $y < x$.

D. Nelson (Washington, D. C.).

Brouwer, L. E. J. On order in the continuum, and the relation of truth to non-contradictority. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 357–358 (1951).

The paper considers four pairs of assertions from the theory of order relations of the intuitionistic continuum [Brouwer, Math. Ann. **95**, 453–472 (1925)]. For each pair, the author states that the assertions are not equivalent while their negations are equivalent. Any one of the pairs serves, in the words of the author, to demonstrate "the theorem of contradictority of the principle of reciprocity of complementarity of species (a fortiori that of contradictority of the equivalence of truth and non-contradictority in mathematics)." *D. Nelson* (Washington, D. C.).

***Van Dantzig, D. Mathématique stable et mathématique affirmative.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 123–135. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

The author describes again [Nederl. Akad. Wetensch., Proc. **50**, 918–929, 1092–1103 = Indagationes Math. **9**, 429–440, 506–517 (1947); these Rev. **9**, 221, 322] two subsystems of intuitionistic mathematics: (1) stable mathematics consisting of those intuitionistic theorems which are equivalent to their double negations; and (2) affirmative mathematics, those theorems containing only the logical operations of conjunction, implication, and universal quantification. The present paper discusses the introduction of negation, disjunction, and bounded existential quantification for a limited class of statements. Several particular results in affirmative mathematics are sketched. These concern systems of linear equations, the existence of a maximum of a continuous function, and the existence of solutions for differential equations $y' = f(x, y)$. *D. Nelson.*

***Heyting, A. L'axiomatique intuitionniste.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 81–86. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

The intuitionist regards a generality statement "For all x, $A(x)$" as an incomplete construction, since it is impossible to carry out as a single construction all of the specific constructions corresponding to the separate statements "$A(o)$" for each object o of an infinite set. It is a recipe, so to speak, for the separate constructions. Similarly an axiomatic system outlines certain constructions with varying explicitness which may be carried out with entities which satisfy the axioms of the system if there are any such. The intuitionist recognizes the practically indispensable role of incomplete constructions in mathematics. However, intuitionistic mathematics is founded upon actual constructions; and the axiomatic systems, logic included, find their interest in affording theorems of a very general nature about the constructed entities of mathematics. This difference in viewpoint accounts for the difference between classical and intuitionistic formal systems. *D. Nelson.*

***Errera, A. Observations sur la communication du Professeur Heyting.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 87–89. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

The author contends that intuitionistic metamathematics is contradictory. In the absence of any well-defined system of metamathematics the contention is vague. Even granting the possibility of correcting this defect, the reviewer questions the assertion, essential to the author's argument, that the principle of excluded middle is intuitionistically neither true nor false. Care should be taken to distinguish various forms which, although classically equivalent, are intuitionistically distinguishable. (For a recent statement of Brouwer's own views on this see Proceedings of the Tenth International Congress of Philosophy, Amsterdam, 1948, North-Holland Publishing Company, Amsterdam, 1949, pp. 1235–1249; these Rev. **10**, 422.) "For all statements A, A is true or false" is intuitionistically false. "For all statements A, it is false, that it is false, that A is true or false" is intuitionistically true. *D. Nelson.*

Grzegorczyk, Andrzej. Undecidability of some topological theories. Fund. Math. **38**, 137–152 (1951).

Verf. geht von der wesentlichen Unentscheidbarkeit (w. U.) einer axiomatisierten Arithmetik (enthaltend Nachfolger, Addition und Multiplikation) aus, für die er auf eine noch nicht veröffentlichte Arbeit von Mostowski, Robinson und Tarski verweist. Hieraus folgt trivial die w. U. einer Theorie T_1 der endlichen Mengen (enthaltend Nachfolger, Addition, Multiplikation und Kardinalzahlgleichkeit). Anschliessend betrachtet Verf. gewisse topologische Raüme, Hüllenalgebren (closure algebras) und ähnliche Strukturen. Zum Beweis der w. U. der zugehörigen Theorien wird gezeigt, dass sich T_1 in diesen interpretieren lässt. *P. Lorenzen* (Bonn).

Kalicki, J. On comparison of finite algebras. Proc. Amer. Math. Soc. **3**, 36–40 (1952).

Im Anschluss an die Problemstellung von G. Birkhoff [Proc. Cambridge Philos. Soc. **31**, 433–454 (1935)] gibt Verf. ein Verfahren an, um zu entscheiden, ob für zwei endliche Algebren dieselben "Gesetze" gelten oder nicht. Das Verfahren ist eine Verallgemeinerung desjenigen des Verf. [J. Symbolic Logic **15**, 174–181 (1950); diese Rev. **12**, 663]. *P. Lorenzen* (Bonn).

Rose, Alan. Eight-valued geometry. Proc. London Math. Soc. (3) **2**, 30–44 (1952).

Eine elementare geometrische Aussage Q, die nur in der elliptischen (parabolischen, hyperbolischen) Geometrie G_e (G_p, G_h) gilt, werde der "Wahrheitswert" a_e (a_p, a_h) zugeordnet. Gilt Q in G_e und G_p, so sei a_{ep} zugeordnet—entsprechend a_{eh}, a_{ph}. Zusammen mit 0 (Q gilt in keiner Geometrie) und 1 (Q gilt in allen Geometrien) bilden diese Werte einen Verband. Für G_e sind die Werte a_e, a_{ep}, a_{eh}, 1 "ausgezeichnet"—entsprechend für G_p, G_h. Verf. stellt in Verallgemeinerung der Untersuchungen von Rosser und Turquette [J. Symbolic Logic **10**, 61–82 (1945); diese Rev. **7**, 185] ein vollständiges Axiomensystem für ein vollständiges System von Wahrheitsfunktionen auf, falls die Werte einen endlichen Verband bilden und mehrere ausgezeichnet sind. *P. Lorenzen* (Bonn).

Bouligand, Georges. Sur l'axiomatique comparée. Revue Sci. **90**, 3–10 (1952).

Cassina, U. Ideografia e logica matematica. Period. Mat. (4) **30**, 65–78 (1952).

Suetuna, Zyoiti. Über die Grundlagen der Mathematik. II. Proc. Japan Acad. **27**, 389–392 (1951).

Continuing his earlier paper [J. Math. Soc. Japan **3**, 59–68 (1951); these Rev. **13**, 310], the author makes some re-

marks about sets and functions. In his opinion, the axiom of choice is valid only for sets which are constituted by an intuition, brought about by action. He makes a difference between a property valid for any given number, and a property valid for all numbers. A function which is defined for any given number is called a pseudo-function; a pseudo-function which is continuous for any given number in an interval is defined for all numbers in that interval.

A. Heyting (Amsterdam).

✓ ***Waismann, Friedrich. Introduction to mathematical thinking. The formation of concepts in modern mathematics.** Translated from the German by Theodore J. Benac. Frederick Ungar Publishing Co., New York, N. Y., 1951. xi+260 pp. **$4.50.**

This is a translation of the author's "Einführung in das mathematische Denken" [Gerold, Vienna, 1947]. It is addressed to the mathematical layman and is concerned largely with laying a rigorous foundation for analysis and the real number system along lines originated by Peano, Cantor, Dedekind, and Weierstrass. The chapter headings are: 1) The various types of numbers; 2) Criticism of the extension of numbers; 3) Arithmetic and geometry; 4) The rigorous construction of the theory of integers; 5) The rational numbers; 6) Foundation of the arithmetic of natural numbers; 7) Rigorous construction of elementary arithmetic; 8) The principle of complete induction; 9) Present status of the investigation of the foundations; 10) Limit and point of accumulation; 11) Operating with sequences, Differential quotient; 12) Remarkable curves; 13) The real numbers; 14) Ultrareal numbers; 15) Complex and hypercomplex numbers; 16) Inventing or discovering.

Considering its scope, this book is remarkably free from errors and misstatements. Its criticisms of the viewpoint of Frege are very severe. There is no treatment of transfinite numbers. Some of the difficult points concerning the real number system are necessarily omitted, since they involve the fact that the real numbers are uncountable, whereas the provable theorems about them are countable. The viewpoint of abstract mathematicians, that mathematics investigates the consequences of quite arbitrary postulate systems without reference to interpretations in the physical world, is not presented. The mathematical thinking described is that of the nineteenth century. *O. Frink* (State College, Pa.).

***Translations from the philosophical writings of Gottlob Frege.** Edited by Peter Geach and Max Black. Philosophical Library, New York, N. Y., 1952. x+244 pp. **$5.75.**

Translations of selected papers and of parts of the Grundgesetze der Arithmetik [Pohle, Jena, 1893] dealing with logic and the foundations of mathematics.

ALGEBRA

Słupecki, J. On the systems of tournaments. Colloquium Math. **2** (1951), 286–290 (1952).

The problem considered is one posed by H. Steinhaus in 1929, namely: given n players with fixed abilities a_i such that $a_1>a_2>\cdots>a_n$ (player a_i is certain to beat player a_j if $j>i$), what is the least number of two-player matches which are sufficient to determine (i) the champion (a_1), (ii) the champion and runner-up (a_1, a_2)? The answer to (i) is $n-1$, to (ii) $n-1+E\log_2(n-1)$ where $E(x)$ is the largest integer less than x. The system of play attaining these is the following; arrange the matches in rounds, each with a maximum number of matches, and after each round eliminate all losers; to determine the runner-up repeat the process on the reduced set of players consisting of those who have lost to the champion. *J. Riordan.*

Nair, K. R. Some three-replicate partially balanced designs. Calcutta Statist. Assoc. Bull. **4**, no. 13, 39–42 (1951).

This paper brings together known examples of partially balanced incomplete block designs with $m=2$ or 3 associate classes, and three replications. A complete enumeration for the case $m=2$ (including a number of new examples) has been recently obtained by W. H. Clatworthy in an unpublished thesis (University of North Carolina).

R. C. Bose (Chapel Hill, N. C.).

Evans, T. A., and Mann, H. B. On simple difference sets. Sankhyā **11**, 357–364 (1951).

A set $a_1, a_2, \cdots, a_k$ of different residues mod v is called a difference set (v, k, λ) if the congruence $a_i-a_j\equiv d \pmod v$ has exactly λ solutions for every $d\not\equiv 0 \pmod v$ [so that $k(k-1)=\lambda(v-1)$]. A simple difference set is a set (v, k, l). By using a number of theorems, some due to Marshall Hall [Duke Math. J. **14**, 1079–1090 (1947); these Rev. **9**, 370], some to Mann in a paper to appear, and some proved here, the authors develop a series of tests which show that for $k+1=n\leqq 1600$, no simple set exists unless n is the power of a prime. *J. Riordan* (New York, N. Y.).

Shanks, E. B. Iterated sums of powers of the binomial coefficients. Amer. Math. Monthly **58**, 404–407 (1951).

The author proves the formula

$$(*)\qquad \binom{n}{i}^k = \sum_{m=1}^{ik-i+1} A(i, k, m)\binom{n+m-1}{ik},$$

where each of the coefficients $A(i, k, m)$ is an integer and depends only on the parameters indicated; he also derives expressions for these coefficients in the form of determinants and proves that $A(i, k, m)=A(i, k, ik-i-m+2)$. He then defines recursively $s(i, k, n, p)=\sum_{j=1}^{n} s(i, k, j, p-1)$, where $s(i, k, n, 0)=\binom{n}{i}^k$ and uses (*) to prove

$$s(i, k, n, p)=\sum_{m=1}^{ik-i+1} A(i, k, m)\binom{n+p+m-1}{ik+p}.$$

For $i=p=1$ we thus get a formula for the sum of the kth powers of the first n integers and the author's result is a two-way generalization of this formula. Other special cases are also of interest. *H. W. Brinkmann* (Swarthmore, Pa.).

Carlitz, Leonard. Note on a paper of Shanks. Amer. Math. Monthly **59**, 239–241 (1952).

Referring to the paper by Shanks whose review precedes this one, the author notes that for $i=1$ the numbers $A(l, k, m)$ were introduced by Euler and that for $i=1$ formula (*) was given by Worpitzky [J. Reine. Angew. Math. **94**, 203–232 (1883)] who also gave a recursion formula for the coefficients in this case. In the present note the author obtains the formula

$$A(i, k, m)=\sum_{r=0}^{m}\binom{ik+1}{r}\binom{m-r+i-1}{i}^k$$

thereby generalizing a result given by Euler for the case $i=1$. He also gives a recursion formula for $A(i,k,m)$ that generalizes Worpitzky's recursion formula; in particular it results that the numbers $A(i,k,m)$ are positive.

H. W. Brinkmann (Swarthmore, Pa.).

Ostrowski, Alexandre. Sur les matrices peu différentes d'une matrice triangulaire. C. R. Acad. Sci. Paris **233**, 1558–1560 (1951).

Let $A=(a_{\mu\nu})$ be a square matrix of order n with $|a_{\mu\nu}|\leqq m$ $(\mu>\nu)$, $|a_{\mu\nu}|\leqq M$ $(\mu<\nu)$, where $0<m<M$. Three results are announced. I. Each eigenvalue of A lies in one of the n closed circles with center $a_{\mu\mu}$ and radius

$$\delta(m,M)=(Mm^{1/n}-mM^{1/n})\cdot(M^{1/n}-m^{1/n})^{-1};$$

the radius $\delta(m,M)$ is the smallest possible. II. In order that no such matrix A with $|a_{\mu\mu}|\geqq 1$ be singular it is necessary and sufficient that

$$(*)\qquad \phi(m,M)=M(1+M)^{-n}-m(1+m)^{-n}>0.$$

When (*) holds,

$$|\det A|\geqq(1+m)^n\,(1+M)^n\,(M-m)^{-1}\,\phi(m,M).$$

III. Let $q=(1+m)\,(1+M)^{-1}$. Then $(M+1)\,(Mq^n-m)\times(M-m)^{-1}$ is $\leqq$ the lower bound of the transformation A of n-space into itself, under the κth power norm $(\kappa=1,2,\infty)$. Theorem II is generalized to let m, M depend on μ.

G. E. Forsythe (Los Angeles, Calif.).

Ostrowski, A. M., and Taussky, Olga. On the variation of the determinant of a positive definite matrix. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 383–385 (1951).

Three results are proved: I. If the matrix $S>0$ (definite) or $S\geqq 0$ (semi-definite), then, for an arbitrary antisymmetric matrix A, $\det(S+A)\geqq\det S$; if $S>0$, strict inequality holds unless $A=0$. II. If H_1 and H_2 are two hermitian matrices and $H_1\geqq 0$, then $|\det(H_1+iH_2)|\geqq\det H_1$; if $H_1>0$ there is strict inequality unless $H_2=0$. III. Let R be a square matrix whose values $\bar{x}'Rx$ lie in the lower half-plane for all unit vectors x. If the hermitian matrix H_1 has characteristic roots $\geqq 0$, then $|\det(H_1+iR)|\geqq\det H_1$. *G. E. Forsythe.*

Roth, William E. The equations $AX-YB=C$ and $AX-XB=C$ in matrices. Proc. Amer. Math. Soc. **3**, 392–396 (1952).

Two theorems are established. I. The necessary and sufficient condition that the equation $AX-YB=C$, where A, B, and C are $m\times r$, $s\times n$, and $m\times n$ matrices, respectively, with elements in a field F, have a solution X, Y of order $r\times n$ and $m\times s$, respectively, and with elements in F is that the matrices

$$\begin{pmatrix}A & C\\ 0 & B\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$$

be equivalent. II. The necessary and sufficient condition that the equation $AX-XB=C$, where A, B, and C are square matrices of order n with elements in F, have a solution X with elements in F is that the matrices displayed above be similar. If $m=n=r=s$, theorem I can be generalised so that F is replaced by the polynomial domain $F[x]$.

D. E. Rutherford (St. Andrews).

Brandt, H. Über das Rechnen mit bilinearen Substitutionen. Jber. Deutsch. Math. Verein. **55**, Abt. 1, 53–67 (1952).

The advantages of employing matrix symbolism and algebra in the study of linear transformations are well known. No correspondingly advantageous symbolism and algebra is available for the study of bilinear transformations because a bilinear transformation W: $x_i=\sum_{j,k}w_{ijk}y_jz_k$ $(i=1,\cdots,m_1;\ j=1,\cdots,m_2;\ k=1,\cdots,m_3)$ evidently requires a 3-way or "space" symbol and there will inevitably be some imperfections in any 2-way or "plane" symbolism which may be employed. Nevertheless, the author proposes a "plane" symbolism, which he has employed to advantage before, and whose usefulness, with the algebra he associates with it, he illustrates in three connections. (1) The derivation and expression of some identities of Frobenius [S.-B. Preuss. Akad. Wiss. **1903**, 504–537, 634–645] in the theory of hypercomplex systems. (2) The calculation of discriminants of simple and semisimple hypercomplex systems [cf. Speiser's chapter in Dickson's Algebren und ihre Zahlentheorie, Füssli, Zürich, 1927]. (3) The author's own studies of bilinear transformations and the composition of quaternary quadratic forms [cf. Math. Z. **20**, 223–230 (1924)]. A brief indication of the symbolism must suffice here. Suppose x, y, and z are column vectors of m_1, m_2, and m_3 elements, resp., and that R, S, and T are matrices of which R is nonsingular while S and T need not be square. If $x=Rx'$, $y=Sy'$, and $z=Tz'$, then the transformation W above becomes W': $x_i'=\sum_{g,h}w'_{igh}y_g'z_h'$. The author writes the relation between W and W', and the transformation W itself, symbolically:

$$W'=R^{-1}W\begin{array}{l}\diagup S\\ \diagdown T\end{array},\qquad x=W\begin{array}{l}\diagup y\\ \diagdown z\end{array}.$$

R. Hull (Lafayette, Ind.).

Kryloff, N. M. Sur l'application des nombres hypercomplexes à la résolution des équations algébriques. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 109–112 (1947). (Russian. French summary)

Writing the roots x_0, x_1, x_2 of a cubic X^3-aX^2+bX-c in the form $x_\alpha=k+mj^\alpha+nj^{2\alpha}$, where j is a primitive cube root of 1, and expressing the elementary symmetric polynomials a, b, c in terms of k, m, n, the author obtains the usual solution of the cubic by radicals. He also finds that $(c^{-1/3}k,\ c^{-1/3}m,\ c^{-1/3}n)$ is a point on the surface ("Appell sphere") $x^3+y^3+z^3-3xyz=1$, and generalizes this fact to certain polynomials of degree $\geqq 3$. *E. R. Kolchin.*

Germay, R. H. Un exemple simple de produit indéfini de facteurs primaires dont les zéros sont les racines d'équations récurrentes. Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 49–54 (1952).

Sherman, S. On the roots of a transcendental equation. J. London Math. Soc. **27**, 364–366 (1952).

Employing a method due to Ansoff and Krumhansl, the author studies the equation (1) $s-a_1-a_2e^{-s}=0$, which has been discussed by Hayes [same J. **25**, 226–232 (1950); these Rev. **12**, 106] in the case that a_1 and a_2 are real. On allowing a_1 and a_2 to be complex, the author obtains the following theorem: If no purely imaginary s is a zero of (1) then a necessary and sufficient condition that all the zeros of (1) lie to the left of $R(s)=0$ is that one of the two following conditions be satisfied: (i) $|a_2|^2<R^2(a_1)$ and $R(a_1)<0$; (ii) $|a_2|^2\geqq R^2(a_1)$ and $N=-\frac{1}{2}[1+\operatorname{sgn}R(a_1)]$. Here the number N is the number of times the function $(-a_2e^{-i\omega})/(i\omega-a_1)$ encircles the point $-1+0i$ in the counterclockwise direction as ω varies from $-\infty$ to $+\infty$. He calls attention to applications to problems of stability of flow for rocket motors.

J. M. Danskin (Santa Monica, Calif.).

Cugiani, Marco. Il teorema di Bézout. Period. Mat. (4) **30**, 98–113 (1952).

Abstract Algebra

★Rennie, Basil C. The theory of lattices. Foister and Jagg, Cambridge, England, 1951. 51 pp. Stiff cover, $1.50; paper cover, $1.00.

Much of this booklet consists of a more detailed exposition of the material in the author's paper of the same title [Proc. London Math. Soc. (2) **52**, 386–400 (1951); these Rev. **13**, 7]. The principal additional topics are indicated by the following. A complete determination is made of the pairs of topologies, among the eight intrinsic topologies of lattices considered, such that closure under the first implies closure under the other. The conditional completion by cuts of a $\cup$-continuous lattice need not be $\cup$-continuous. Properties of these topologies in Boolean algebras are considered. The lattice of measurable subsets of the unit interval, modulo sets of measure 0, is not a Hausdorff space in the interval topology; see problems 25 and 76 of G. Birkhoff's "Lattice Theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. **10**, 673]. If L is the lattice of closed sets of a topological space T, then L is bounded, conditionally complete, distributive, $\cup$-continuous but not necessarily $\cap$-continuous, and if b is the inf of the set X then $b\cup c$ is the inf of the set of all $c\cup x$, $x\varepsilon X$. In T, every non-empty open set contains a non-empty closed set if and only if in L, $y<1$ implies the existence of $x\neq 0$ with $x\cap y=0$; then a subset of T is both open and closed if and only if it has a complement.

A lattice group is necessarily a $\cup$- and $\cap$-continuous distributive lattice. Let λ be a measure and M the λ-measurable sets of the real line modulo sets of measure 0. Properties of M as a lattice are studied; e.g. it is a $\cup$- and $\cap$-continuous Boolean algebra. $L(x, y)$ is the value of θ for which

$$\lambda([\text{interval } -1/\theta, 1/\theta]\cap[\text{symmetric difference of } x, y])=\theta;$$

it is a metric, with topology equivalent to L-, order-, and star-topologies. $S(x, y)$ is the sup for real θ of

$$\lambda([\text{symmetric difference of } x, y]\cap[\text{interval } \theta, \theta+1]).$$

M is complete in the L and S metrics. For $x\varepsilon M$, denote by $x(\theta)$ the set x moved a distance θ to the right. Then x is called L-almost-periodic (L-a.p.) if the L-closure of the set of all $x(\theta)$, θ real, is L-compact. The set of L-a.p. elements of M is a complemented sublattice. Periodicity implies S-a.p., which implies L-a.p. *P. M. Whitman.*

Sorkin, Yu. I. Free unions of lattices. Mat. Sbornik N.S. **30**(72), 677–694 (1952). (Russian)

Given disjoint lattices S_α ($\alpha\varepsilon M$), their free union $S=\coprod^*_{\alpha\varepsilon M}S_\alpha$ is the most general lattice containing the S_α and preserving meets and joins, and is obtained in accordance with Dilworth's inductive procedure [Trans. Amer. Math. Soc. **57**, 123–154 (1945); these Rev. **7**, 1]. If there is a homomorphism ϕ_α of each S_α onto a lattice T then there is a homomorphism of S onto T coinciding with ϕ_α on each S_α; a similar theorem holds for isotone mappings in place of homomorphisms. The operation of taking the direct union is associative.

In the remainder of the paper it is assumed that each S_α is a chain; results and methods parallel those of the reviewer [Ann. of Math. **42**, 325–330 (1941); **43**, 104–115 (1942); these Rev. **2**, 244; **3**, 261] for the case of single-element chains (S a free lattice). In particular, $A\geqq B$ in S if and only if this holds in some S_α or $A=A_1\cup A_2$ with some $A_i\geqq B$ or $A=A_1\cap A_2$ with each $A_i\geqq B$ or dually. The set union of the S_α is the only irreducible set of generators of S. Given any word in S, there is a unique (within associativity and commutativity) word of minimum rank equal to it. The preceeding statements in this paragraph are not generally true if the S_α are not chains. The group of automorphisms of S is the direct product of symmetries of disjoint subsets of M.

Hasse diagrams are obtained for the free union of a single element with a chain of 1, 2, or 3 elements [see G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, pp. 66–68; these Rev. **10**, 673]; all other free unions of chains are infinite. *P. M. Whitman.*

Hashimoto, Junji. On a lattice with a valuation. Proc. Amer. Math. Soc. **3**, 1–2 (1952).

A real-valued function $v(x)$ on a lattice L is called a distributive valuation if it satisfies, for $x, y, z\varepsilon L$, the equation

$$2\{v(x\cup y\cup z)-v(x\cap y\cap z)\}=\{v(x\cup y)+v(y\cup z)+v(z\cup x)\}\\-\{v(x\cap y)+v(y\cap z)+v(z\cap x)\}.$$

The main result of the present paper is given by the following theorem: A necessary and sufficient condition for a lattice L to be distributive consists in the property that there exists, for every $x<y$ in L, a distributive valuation, which is defined on L and is not constant on the interval $[x, y]$. The proof is based upon properties of certain five-element sublattices in modular and not modular lattices. *O. Borůvka* (Brno).

Trevisan, Giorgio. Sulla distributività delle strutture che posseggono una valutazione distributiva. Rend. Sem. Mat. Univ. Padova **20**, 396–400 (1951).

A simple solution is given to Problem 71 of the reviewer's "Lattice theory", 2d ed. [Amer. Math. Soc., New York, 1948; these Rev. **10**, 673]; this has also been solved by J. Hashimoto [see the preceding review]. *G. Birkhoff.*

Petresco, Julian. Théorie relative des chaînes. I. Conformisme et correspondance. C. R. Acad. Sci. Paris **235**, 226–228 (1952).

Consider two ascending chains $\alpha=\{A_i\}$, $\beta=\{B_j\}$ with the same extremities. Write

$$A_{ij}=A_{i-1}\cup(A_i\cap B_j),\quad B_{ji}=B_{j-1}\cup(B_j\cap A_i);$$

α and β are called $\cap$-cosaturated if $\alpha=\{A_{ij}\}$, $\beta=\{B_{ji}\}$. The author states various theorems of the Jordan type, without consideration of isomorphisms. For example: for $\cap$-cosaturated chains, the following are equivalent: (a) for all i, j, $A_{i,j-1}=A_{ij}$ if and only if $B_{j,i-1}=B_{ji}$ ("$Z\cap$-conformes"); (b) there is a one-to-one correspondence between the prime intervals in α and β such that

$$A_{i-1}\cap B_{j-1}=A_{i-1}\cap B_j=A_i\cap B_{j-1}\neq A_i\cap B_j$$

or else $A_{i-1}=A_i$, $B_{j-1}=B_j$ ("$J\cap$-correspondantes"); (c) the correspondence of α into β such that $A_{i,j-1}=A_{i-1}$ and $A_{ij}=A_i$, and the similar correspondence of β into α, are one-to-one ("canoniquement $\cap$-correspondantes"). *P. M. Whitman* (Silver Spring, Md.).

Šul'gelfer, E. G. Decomposition into prime factors in structures with multiplication. Ukrain. Mat. Žurnal **2**, no. 3, 100–114 (1950). (Russian)

A structure with multiplication (or residuated lattice) is a lattice S with an auxiliary associative multiplication and

appropriate axioms linking the operations. Elements e and 0 in the lattice sense are assumed. S is said to have a correct arithmetic if every element other than 0 and e is a product of primes, unique up to order, and if $a \leq b$ implies that a is both a left and right multiple of b. This is shown to be equivalent to the following six conditions. (1) e is a multiplicative unit; (2) maximal elements commute; (3) the ascending chain condition; (4) prime elements are maximal; (5) non-zero powers of a maximal prime are distinct; (6) there is no element strictly between the nth and $(n+1)$th powers of a maximal prime. The proof is lengthy but proceeds along familiar lines. In §4 examples show the independence of the six conditions. The result is compared with earlier theorems of this kind for rings. An example shows that a non-commutative non-simple ring, with a correct arithmetic for two-sided ideals, need not have a unit element. Finally the author takes up his own multiplicative theory of quasi-ideals [Doklady Akad. Nauk SSSR **64**, 633–636 (1949); these Rev. **10**, 502]. It is shown that a correct arithmetic for ordinary ideals implies one for quasi-ideals; but the ring of even integers is a counter-example to the converse. *I. Kaplansky* (Chicago, Ill.).

Dubreil-Jacotin, Marie-Louise. Théorèmes de décomposition dans certains treillis et demi-groupes réticulés sans condition de chaîne. C. R. Acad. Sci. Paris **234**, 2415–2416 (1952).

An element $x \neq I$ in a lattice L is called t-$\cap$-irreducible if the section of L above x is linearly ordered. Let $a \equiv b$ (F) mean that $a \cup x = I$ if and only if $b \cup x = I$. A lattice D is called a generalized Dedekind lattice if every $a \in D$, $a \neq I$, is a finite intersection of t-$\cap$-irreducible elements and any two irreducible representations of the same element have equal numbers of factors, which are pairwise congruent mod F. The author considers uniqueness of intersection-decomposition in lattices which satisfy these and certain additional conditions. *P. M. Whitman* (Silver Spring, Md.).

Emerson, Marion Preston. Dualities of modular lattices. Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. ii+1+i pp.

Duschek, Adalbert. Die Algebra der elektrischen Schaltungen. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 115–134 (1951).

Essentially an abridgement of earlier work [Plechl and Duschek, Österreich. Ing.-Arch. **1**, 203–230 (1946); these Rev. **9**, 560; and Duschek, Monatsh. Math. **52**, 89–123 (1948); these Rev. **10**, 96]. The author continues to be unacquainted with the earlier results of Shannon along the same lines [Trans. Amer. Inst. Elec. Engrs. **57**, 713–723 (1938)]. *S. Sherman* (Sherman Oaks, Calif.).

Thierrin, Gabriel. Sur les homogroupes. C. R. Acad. Sci. Paris **234**, 1519–1521 (1952).

A semi-group H is termed a homogroup if it contains an idempotent element e such that 1) $ex = xe$ for all x in H, 2) given x in H, there exists x' in H such that $xx' = e$. Then e is unique and is termed the unity of H. Also, $N = eH$ is a group homomorphic to H under the mapping $x \rightarrow xe$ and is an ideal of H. Every finite cyclic semi-group is a homogroup. Further, a ring which is a homogroup with respect to multiplication is a field. *D. Rees* (Cambridge, England).

Thierrin, Gabriel. Sur les homodomaines et les homocorps. C. R. Acad. Sci. Paris **234**, 1595–1597 (1952).

A domain H with two operations, addition and multiplication, is termed a homodomain if it has the following properties: 1) it is a homogroup [see the preceding review] with respect to addition; 2) it is a semi-group with respect to multiplication; 3) multiplication is distributive over addition; 4) the unity 0 of H with respect to addition satisfies $x0 = 0x = 0$ for all x in H. It is shown that the set of elements of the form $x+0$ form a ring homomorphic to H. Further, a homodomain H is termed a homofield if there is an element x such that $x+0 \neq 0$ and $H-0$ is a homogroup with respect to multiplication. In this case $H+0$ is a field. If $H-0$ is a group H is a field. Finally it is shown that there exist finite homofields in which multiplication is not commutative. *D. Rees* (Cambridge, England).

Dean, Burton Victor. Near rings and their isotopes. Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. ii+3+i pp.

Almeida Costa, A. Three lectures on the general theory of rings. Anais Fac. Ci. Porto **36**, 65–83 (1952). (Portuguese)

This is an expository account of the Brown-McCoy radical, the maximal regular ideal, and the anti-radical (or socle) of a ring. *I. Kaplansky* (Chicago, Ill.).

Wolfson, Kenneth Graham. An ideal-theoretic characterization of the ring of all linear transformations. Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. ii+2+i pp.

Szélpál, I. Über gewisse Erweiterungen von periodischen Ringen. Publ. Math. Debrecen **2**, 134–136 (1951).

It is proved that a periodic ring R (one having all its elements of finite characteristic) is an ideal in every periodic extension-ring if and only if R is a zero ring whose additive group is a direct sum of Prüfer p^{∞}-groups. *R. E. Johnson* (New Haven, Conn.).

Nagata, Masayoshi. On the theory of radicals in a ring. J. Math. Soc. Japan **3**, 330–344 (1951).

This paper, partly expository in character, gives a unified treatment of various known radicals of a ring, and introduces some new ones. In this review only a few points can be mentioned. The usual definition of an m-system is modified as follows. A subset M of a ring R is called an m-system if $a, b \in M$ implies that there exist elements x, y, z in R (or in the ring R with a unit element adjoined) such that $xaybz \in M$. It is then easy to prove that the lower radical of Baer [Amer. J. Math. **65**, 537–568 (1943); these Rev. **5**, 88] coincides with the radical of the reviewer [ibid. **71**, 823–833 (1949); these Rev. **11**, 311]. This result has been previously obtained by Levitzki [ibid. **73**, 25–29 (1951); these Rev. **12**, 474]. A ring R is called e-primitive if every nonzero ideal in R contains a nonzero idempotent. An ideal $\mathfrak{a}$ is e-primitive if $R/\mathfrak{a}$ is e-primitive. The e-radical of R is the intersection of all e-primitive ideals in R. This coincides with the F-radical of Brown and McCoy [ibid. **69**, 46–58 (1947); these Rev. **8**, 433] if we set $F(a) = (a^2 - a)$. The author discusses some properties of the e-radical and its relation to the radicals of Azumaya [Jap. J. Math. **19**, no. 4, 525–547 (1948); these Rev. **11**, 316] and Jacobson [Amer. J. Math. **67**, 300–320 (1945); these Rev. **7**, 2]. *N. H. McCoy.*

Andrunakievič, V. A. On the determination of the radical of a ring. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 217–224 (1952). (Russian)

The author develops the theory of the Brown-McCoy radical from the point of view of quasi-ideals (a quasi-ideal is most readily visualized in a ring with unit, where it is the result of adding 1 to the elements of an ideal). The radical of R turns out to be the set of all elements x such that the two-sided quasi-ideal generated by x is all of R.

I. Kaplansky (Chicago, Ill.).

Fuchs, L. A remark on the Jacobson radical. Acta Sci. Math. Szeged **14**, 167–168 (1952).

The radical of a ring R with left unit is the analog of the Φ-subgroup in group theory: the set of all elements of R which may be omitted from each generating system of the right R-module R. *D. Zelinsky* (Evanston, Ill.).

Krull, Wolfgang. Jacobsonsche Ringe, Hilbertscher Nullstellensatz, Dimensionstheorie. Math. Z. **54**, 354–387 (1951).

The principal results of this paper are announced with sketches of proofs in Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, v. 2, pp. 56–64 [Amer. Math. Soc., Providence, 1952; these Rev. **13**, 526].

A Jacobson ring is a commutative ring with unit in which every prime ideal is the intersection of maximal ideals. Theorem 1: If R is Jacobson, then so is every finite ring extension $S=R[a_1, \cdots, a_n]$, and if h is a homomorphism of S onto a field then $h(R)$ is also a field and $h(S)$ is algebraic over $h(R)$. (This is the generalized Nullstellensatz.) Theorem 2: If $M\cap R$ is maximal for every maximal ideal M in the polynomial ring $R[x]$, then R is Jacobson. These results have also been obtained by O. Goldman, who used the term Hilbert rings [Math. Z. **54**, 136–140 (1951); these Rev. **13**, 427]. Theorem 4: If R is a field and S a polynomial ring over R in an infinite set A of indeterminates, then the conclusion of Theorem 1 is true if and only if the cardinality of A is less than that of R.

The remainder of the paper is concerned with Noetherian rings and their dimension theory. The main result here is Theorem 18: Let R be Noetherian and Jacobson, let $R'=R[x_1, \cdots, x_n]$ be a polynomial ring, let P' and P be prime ideals of dimensions d' and d in R' and R respectively; if $P=P'\cap R$, then $d'-d$ is the transcendence degree of R'/P' over R/P. This generalizes the classical result where R is a field. Suppose a Noetherian, Jacobson ring R has the following property: If P is prime in R, then all maximal chains of prime ideals beginning with P have the same length. The author raises the question whether this property is inherited by polynomial rings over R, and he discusses the relationship between this question and some other difficult problems in commutative ring theory. *I. S. Cohen.*

Nakayama, Tadasi. Automorphisms of simple, complete primitive, and directly indecomposable rings. Ann. of Math. (2) **55**, 538–551 (1952).

Theorem 1: If A is simple with unit element and minimal condition, if $A\supset B\supset S$ where B is simple and $S=V(T)$ (commuter) where T is a simple (finite) subalgebra over the center of A, and if α is an isomorphism of B into A which leaves S elementwise fixed, then α can be extended to an automorphism of A which is in this case necessarily inner. Theorem 2: The assumption $S=V(T)$ in Theorem 1 may be replaced by the assumption that S is weakly normal (=galoisien in the sense of Dieudonné [Comment. Math. Helv. **21**, 154–184 (1948); these Rev. **9**, 563]) and $V(B)$ and $V(B^\alpha)$ are simple or merely directly indecomposable. A similar theorem holds when A is completely primitive (N. Jacobson [Ann. of Math. **48**, 8–21 (1947); these Rev. **8**, 433]; ="closed irreducible" of Nakayama and Azumaya [ibid. **48**, 949–965 (1947); these Rev. **9**, 563]). There is an application to outer Galois theory which generalizes N. Jacobson's [unpublished] generalization of a theorem of H. Cartan [Ann. Sci. École Norm. Sup. (3) **64**, 59–77 (1947); these Rev. **9**, 325] (volume number and date are misprinted in Nakayama's reference to this paper). *G. Whaples.*

Allen, H. S. Groups of automorphisms on a module. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 253–254 (1952).

Let A be a ring and E be a left A-module. For any submodule M of E, let G_M be the group of automorphisms of E leaving M invariant and g_M be the subgroup of G_M leaving M elementwise invariant. If E is the supplementary sum of M and N, then it is proved that $G_M\cdot G_N=g_M\cdot g_N$.

R. E. Johnson (Northhampton, Mass.).

Foster, Alfred L. p^k-rings and ring-logics. Ann. Scuola Norm. Super. Pisa (3) **5**, 279–300 (1951).

L'auteur considère les algèbres commutatives P sur un corps fini F_q ($q=p^k$, p premier), ayant un élément unité et telles que $x^q=x$ pour tout $x\varepsilon P$. La théorie générale des algèbres (Jacobson-Chevalley) montre aussitôt que P est une sous-algèbre d'un produit dont tous les facteurs sont identiques à F_q; en d'autres termes, P peut être considérée comme algèbre de fonctions à valeurs dans F_q, définies dans un certain ensemble I; si on écrit une telle fonction $x=\sum_{\mu\varepsilon F_q}\mu\cdot x_\mu$, où x_μ est la fonction caractéristique de l'ensemble des éléments de I où x prend la valeur μ, les x_μ sont dans P. L'auteur définit une permutation $x\rightarrow x^\frown$ de P en écrivant les éléments $\neq 0$ de F_q sous la forme ξ^h (ξ élément primitif), et en posant $(x^\frown)_\mu=x_{\pi(\mu)}$, où π est la permutation circulaire de F_q telle que $\pi(\mu)=\mu/\xi$ pour $\mu\neq 0$ et $\mu\neq 1$, $\pi(0)=\xi^{q-2}$, $\pi(1)=0$. Il transforme alors par cette permutation les lois de composition de P, et montre entre autres que $x+y$ peut s'exprimer à l'aide de combinaisons des opérations de produit et de l'opération $^\frown$. L'article contient (p. 282) une curieuse assertion relative au produit direct $F_2\times F_3$; selon l'auteur, cet anneau contiendrait un sous-corps isomorphe à F_6 (et qui ne pourrait donc être que l'anneau tout entier)!

J. Dieudonné (Ann Arbor, Mich.).

Szekeres, G. A canonical basis for the ideals of a polynomial domain. Amer. Math. Monthly **59**, 379–386 (1952).

It is shown that every ideal M of $R[x]$, where R is the domain of rational integers, has a basis $(g_0(x), g_1(x), \cdots, g_m(x))$ with the following properties

(1) $$g_0(x)=q_1q_2\cdots q_m,$$

(2) $$q_kg_k(x)=xg_{k-1}(x)+\sum_{i=0}^{k-1}b_{ki}g_i(x),$$

(3) $$q_k>0,\quad 0\leqq b_{ki}<q_k\quad (0<k\leqq m;\ 0\leqq i<k).$$

The integer m is called the "degree" of M. The integers q_k, b_{ki} are uniquely determined by M. There is a 1-1 correspondence between the ideals of $R[x]$ and the system of integers (3). As an application the author determines the

number of ideal divisors of degree m which contain the constant integer c. Finally, there is a discussion of the case in which R is a domain of integrity in which all ideals are principal. The previous argument applies except that a substitute is required for conditions (3); in some cases this can be done in a simple way. *L. Carlitz.*

Nagata, Masayoshi. On Krull's conjecture concerning valuation rings. Nagoya Math. J. **4**, 29–33 (1952).

It is shown that a field K contains a completely integrally closed primary (existence of at most one proper prime ideal) integral domain which is not a valuation ring of a rank one valuation if and only if (i) it is not absolutely algebraic for characteristic 0, or (ii) it has at least transcendence degree 2 over its prime field for non-zero characteristic. The proof is based on the transitivity of complete integral closure for algebraic extensions and the structure of the valuation rings in such as quotient rings of maximal ideals together with an example of a completely integrally closed primary ring $\mathfrak{o}$ which is not a valuation ring (counterexample to a conjecture of Krull). The counterexample uses a rational function field with an algebraically closed coefficient field F which admits a rank one valuation w whose value group is not the additive group of real numbers; $\mathfrak{o}$ is given as the intersection of valuation rings in $F(x)$ which belong to certain specified extensions of w by means of generalized Rella-Ostrowski constructions. *O. F. G. Schilling* (Chicago, Ill.).

Schmidt, Jürgen. Über die Rolle der transfiniten Schlussweisen in einer allgemeinen Idealtheorie. Math. Nachr. **7**, 165–182 (1952).

This is a study of what the author calls the hull-structure (Hüllenstruktur) of an algebra; it uses, at least for some of its conclusions, transfinite set-theoretic methods including the axiom of choice. A hull-structure is a system of sets with a closure operation in the sense of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948, p. 49; these Rev. **10**, 673]. An algebra, in the author's sense, is a system consisting of a set E and a family $\mathfrak{R}$ of relations defined over E. A subset M of E is closed with respect to a relation $R(x_1, \cdots, x_m, y)$ of $\mathfrak{R}$ if Y is in M whenever $x_1, \cdots, x_m$ are in M and $R(x_1, \cdots, x_m, y)$ holds. The hull-structure $\mathfrak{H}$ consisting of the sets closed under all R in $\mathfrak{R}$ simultaneously is called the hull-structure of the algebra. If the closed sets of $\mathfrak{H}$ have a minimum, the author defines a hull system $\mathfrak{H}^*$ in which this minimal closed set is the closure of 0; otherwise he takes $\mathfrak{H}^*$ to be $\mathfrak{H}$. The closed sets of $\mathfrak{H}^*$ are called ideals; this reduces to the usual notions for a ring if a ring is regarded as a group with operators. The author defines an algebraic hull system as one such that the closure of any set is the sum of the closures of its finite subsets; he shows that $\mathfrak{H}$ and $\mathfrak{H}^*$ are algebraic and conversely any algebraic hull-structure is either the $\mathfrak{H}$ or the $\mathfrak{H}^*$ of some algebra. He defines a family $\mathfrak{M}$ of sets as inductive if, given any chain of sets of $\mathfrak{M}$, the union of the chain is also a set of $\mathfrak{M}$. His principal theorem is that the closed sets of an algebraic hull-structure form an inductive family and conversely. This entails as corollaries certain theorems about determination of ideals by irreducible ideals, maximal ideals, finite generating sets, units, etc. which are credited to McCoy, Fuchs, Lindenbaum, and Krull, but were proved by these authors under somewhat more specialized assumptions. If assumptions regarding set-theoretic addition are made concerning a hull-structure, so that it is converted into a topological space, then it is algebraic if and only if it is completely additive; so that the more interesting topological spaces are transcendental. The paper concludes with theorems expressing connection with the notions of end (Ende), independent family of sets, sets of finite character, basis, etc. *H. B. Curry* (State College, Pa.).

Dynkin, E. B. Semisimple subalgebras of semisimple Lie algebras. Mat. Sbornik N.S. **30**(72), 349–462 (3 plates) (1952). (Russian)

This is a very thorough study of the subject indicated by the title. Pertinent earlier references are the following, the first being by Malcev and all the rest by the author: (1) Izvestiya Akad. Nauk SSSR. Ser. Mat. **8**, 143–174 (1944) = Amer. Math. Soc. Translation no. **33**; these Rev. **6**, 146; (2)–(7) Doklady Akad. Nauk SSSR **71**, 221–224 (1950); **73**, 877–880 (1950); **75**, 333–336 (1950); **76**, 629–632 (1951); **78**, 5–7 (1951); **81**, 987–990 (1951); these Rev. **11**, 492; **12**, 238, 589, 585; **13**, 10, 527; (8) Trudy Moskov. Mat. Obšč. **1**, 39–166 (1952).

When we are given a subalgebra G' of a Lie algebra G, we are equally well given a representation of G' in G. Thus one may say that the problem is to study the representations of a simple Lie algebra G' in a simple Lie algebra G, the appropriate classification being under inner automorphisms of G. When G is A_n, this is the classical theory of representations, fully developed by Cartan and Weyl. When G is B_n, C_n, or D_n the topic is orthogonal or symplectic representations, and was taken up by Malcev in reference (1). Malcev also studied representations in the exceptional group G_2 and, in part, F_4. The author completes the work on the exceptional groups and studies numerous related questions. Much of the work is presented in 42 tables, which occupy about half the paper.

We shall summarize the paper by chapters. In Chapter I certain basic tools are introduced. In order to effect a link with ordinary representation theory, two representations f_1 and f_2 of G' in G are defined to be linearly equivalent if for every linear representation ϕ of G, ϕf_1 and ϕf_2 are equivalent in the ordinary sense. A corresponding notion of linear conjugacy of subalgebras is introduced, and conditions for it and for linear equivalence are studied. In §2 the concept of index is introduced. If f is a representation of G' in G, then the Cartan inner product $(f(x), f(y))$ is an integral multiple of the inner product in G'; this multiple is called the index. Connected with this is the concept of an integral representation or subalgebra; the definition is too elaborate to reproduce here. Chapter II is devoted to regular subalgebras, R-subalgebras and S-subalgebras. Full details are given concerning the results on regular subalgebras announced in reference (3). A subset M of G is called an R-system if it can be embedded in a proper regular subalgebra; otherwise it is an S-system. The R-property is closely connected with reducibility. For example, Theorem 7.1 asserts that in any faithful linear representation of a semi-simple Lie algebra, an R-system is mapped on a reducible set of matrices. Conversely, in the classical algebras any reducible subset is R. Another result is that every S-subalgebra is integral. Chapter III is devoted entirely to 3-dimensional subalgebras. The principal results are assembled in tables 16–20, which give the 3-dimensional simple subalgebras of the exceptional algebras. Chapter IV studies the simple subalgebras of rank >1 (the case of rank 1 being that of Chapter III). The subalgebras of the exceptional algebras are tabulated in table 25 on two large inserted sheets. In Chapter V the author turns to the classification of the S-subalgebras (not

necessarily simple) of the exceptional algebras; they appear in table 39. The final chapter is brief and takes up two topics. The first adds some remarks on maximal subalgebras to his previous work in reference (8). The second is the classification of subalgebras irreducible in some linear representation of the large algebra. The classical cases were handled in reference (8) and the exceptional ones appear in tables 40–42. *I. Kaplansky* (Chicago, Ill.).

Thrall, R. M. On a Galois connection between algebras of linear transformations and lattices of subspaces of a vector space. Canadian J. Math. 4, 227–239 (1952).

Let V be a vector space over a field k, and $\mathfrak{N}$ be the lattice of all subspaces of V. With a sublattice $\mathfrak{L}$ of $\mathfrak{N}$, let $\mathfrak{L}^+$ be the algebra of all linear transformations of V which map every element of $\mathfrak{L}$ into itself. With an algebra $\mathfrak{A}$ of linear transformations of V, let $\mathfrak{A}^*$ be the lattice of $\mathfrak{A}$-subspaces of V. Two necessary conditions for $\mathfrak{L}$ to be closed in this Galois correspondence (i.e. $\mathfrak{L}^{+*}=\mathfrak{L}$) are obtained, among others. Thus, a projectivity of two quotients S/R and T/R gives rise to a natural isomorphism σ of the residue-spaces $S-R$ and $T-R$. If $a\epsilon k$, the elements of $S+T$ lying in cosets $(s+R)+\sigma(s+R)a$ (s running over S) modulo R form a space Q_a. $\mathfrak{L}$ is called projectively closed, in $\mathfrak{N}$, if $\mathfrak{L}$ contains every such space Q_a obtained from quotients S/R, T/R in $\mathfrak{L}$ (and $a\epsilon k$). Then, if $\mathfrak{L}$ is closed, $\mathfrak{L}$ is projectively closed. Secondly, if $\mathfrak{L}=\mathfrak{A}^*$ with a cleft algebra $\mathfrak{A}$, then $\mathfrak{L}$ possesses the relative imbedding property, i.e. $\mathfrak{L}$ is contained in a complemented sublattice of $\mathfrak{N}$ having the same dimensionality and projective structure constants [Thrall, Proc. Amer. Math. Soc. 2, 146–152 (1951); these Rev. 12, 795] as $\mathfrak{L}$. On the other hand, if $\mathfrak{L}$ is distributive, $\mathfrak{L}$ is closed. Further, $\mathfrak{A}$ is closed (i.e. $\mathfrak{A}^{*+}=\mathfrak{A}$) with $\mathfrak{A}^*$ distributive if and only if the irreducible constituents of $\mathfrak{A}$ are inequivalent total matrix algebras. With a distributive $\mathfrak{L}$, algebras $\mathfrak{A}$ satisfying $\mathfrak{A}^*=\mathfrak{L}$ are studied, in terms of join-irreducible elements of $\mathfrak{L}$. *T. Nakayama* (Nagoya).

Tate, John. Genus change in inseparable extensions of function fields. Proc. Amer. Math. Soc. 3, 400–406 (1952).

Let $K=k(\alpha)$ be an inseparable extension of degree p over a field k with characteristic $p\neq 0$. For

$$\xi=x_0+x_1\alpha+\cdots+x_{p-1}\alpha^{p-1}\quad (x_i\epsilon k),$$

set $S_\alpha(\xi)=x_{p-1}$. With a second generator β of K/k, it is shown that $S_\beta(\xi)=S_\alpha(\xi(D_\alpha\beta)^{1-p})$ (for all $\xi\epsilon K$), where D_α is the derivation in K, vanishing on k, with respect to α. If k is in particular an algebraic function field of one variable, k_0 and K_0 are the constant fields of k and K, and if S_0 is any nontrivial k_0-linear map of K_0 into k_0, then for any differential ω of k there exists uniquely a differential Ω of K such that $S_0(\Omega(\mathfrak{X}))=\omega(S_\alpha(\mathfrak{X}))$. This construction is combined with the quasi-different of K/k with respect to the quasi-trace S_α, and further with the above result, to give

$$2G-2=p^{1-n}(2g-2)+(1-p)\sum_{\mathfrak{P}}\nu_{\mathfrak{P}}(D_{\tau_{\mathfrak{P}}}\alpha)\deg\mathfrak{P},$$

where G, g are the genera of K, k and $p^n=(K_0:k_0)$ while $\nu_{\mathfrak{P}}$ is the ordinal number function for a place $\mathfrak{P}$ in K and $\tau_{\mathfrak{P}}$ is a generator of the (local) integral domain of $K_{\mathfrak{P}}$ over that of $k_{\mathfrak{p}}$ ($\mathfrak{p}$ being the place of k below $\mathfrak{P}$). It follows in particular that the genus change in totally inseparable extension is always a multiple of $\frac{1}{2}(p-1)$. It is shown how the above numbers $\nu_{\mathfrak{P}}(D_{\tau_{\mathfrak{P}}}\alpha)\deg\mathfrak{P}$ may be computed in the ground field in terms of $\alpha^p\epsilon k$. *T. Nakayama* (Nagoya).

***Faddeev, D. K. Simple algebras over a field of algebraic functions of one variable.** Trudy Mat. Inst. Steklov., v. 38, pp. 321–344. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The constant field k_0 for the algebraic functions is always an algebraic number field. The problem is to classify simple algebras over $k=k_0(x,y)$, where x is an indeterminate and y is algebraic over $k_0(x)$. Any simple algebra over k_0 can be lifted up to one over k; the result is called a numerical algebra. The principle of classification is the following: two algebras over k are similar in the wide sense if one is similar in the ordinary sense to the product of the other by a numerical algebra.

After some expository material, the author takes up the local theory. Let A be a central division algebra over the field $k_0(\pi)$ of formal power series in π. The center Z of the inertial algebra turns out to be cyclic over k_0. If Π is an element of minimal positive value in A, then the inner automorphism by Π induces an automorphism ζ of Z. The "cyclic pair" (Z,ζ) is shown to be a complete set of invariants for A under similarity in the wide sense. The theory extends to the simple case.

Let now A be a global central simple algebra. For every divisor p of k we have a local algebra A_p over the local field k_p, which is a field of power series over the inertial field k_1 of p. The algebra A_p has as its invariant the cyclic pair defined above, and these cyclic pairs satisfy a product formula

$$\prod_p N_{k_1/k_0}(Z_p,\zeta_p)=1.$$

The definition of the norm and product of cyclic pairs is too lengthy to reproduce here; it uses Galois theory, character groups and the transfer (Verlagerung) of a group into a subgroup. Finally, k is specialized to the field $k_0(x)$ of rational functions, and it is shown that the local invariants of A characterize A up to similarity in the wide sense. Moreover there exists an algebra for any set of invariants satisfying the product formula. *I. Kaplansky.*

Delone, B. N., Kuroš, A. G., Kolmogorov, A. N., Markov, A. A., Gel'fond, A. O., Meiman, N. N., Sanov, I. N., and Vilenkin, N. Ya. Paths of development of algebra. Uspehi Matem. Nauk (N.S.) 7, no. 3(49), 155–178 (1952). (Russian)

Report of a discussion which took place at a conference on algebra and the theory of numbers held in Moscow, 6–12 September, 1951.

Theory of Groups

Deuring, Max. Die Gruppentheorie. Akad. Wiss. Mainz. Jahrbuch 1951, 270–276 (1951).

"Die Absicht dieses Vortrages ist, an einem Beispiel die moderne mathematische Begriffsbildung und Denkweise aufzuzeigen."

Bruck, R. H. Pseudo-automorphisms and Moufang loops. Proc. Amer. Math. Soc. 3, 66–72 (1952).

Die bisherigen Untersuchungen über loops, in denen die Identität $xy\cdot zx=x(yz\cdot x)$ gilt—Moufang loops genannt und im Folgenden mit G bezeichnet—[s. insbesondere Bruck, Trans. Amer. Math. Soc. 60, 245–354 (1946); Proc. Amer. Math. Soc. 2, 144–145 (1951); Bull. Amer. Math. Soc. 57, 11–26 (1951); Bruck und Kleinfeld, Proc. Amer. Math. Soc.

2, 878–890 (1951); diese Rev. 8, 134; 13, 9; 12, 585; 13, 526] werden vertieft und die Beweise vereinfacht, wodurch bemerkenswerterweise auch für nichtkommutative loops Induktionsschlüsse überflüssig werden. Die vorliegende Note kann als eine systematische Einführung in die Theorie der Moufang loops angesehen werden. Der Associator (x, y, z) ist definiert durch $xy\cdot z=(x\cdot yz)(x, y, z)$, der Commutator (x, y) durch $xy=(yx)(x, y)$. In G gelten folgende Associativitätstheoreme. 1) Seien A, B, C Untermengen von G, so dass

$$(A, A, G)=(B, B, G)=(C, C, G)=(A, B, C)=1$$

ist; dann ist $A\cup B\cup C$ in einem associativen Subloop von G enthalten. 2) Sei A ein associativer Subloop von G, B eine Untermenge von G mit $(A, A, B)=(B, B, G)=1$, dann ist $A\cup B$ in einem associativen Subloop von G enthalten. 3) In G gelte $(G, G, (G, G))=1$; dann ist jede maximale associative Untermenge von G ein maximaler associativer Subloop von G. Ferner beweist Verf. einige Beklammerungsrelationen in Produkten aus vier Symbolen, falls je drei von ihnen sich associativ multiplizieren. Das grundlegende Hilfsmittel bei diesen Strukturuntersuchungen ist der Begriff des Pseudo-Automorphismus eines loops: eine Permutation U des loop G heisst ein Pseudo-Automorphismus, falls mindestens ein Element u, der sog. Begleiter von U, in G existiert, so dass für alle x, y aus G gilt $(xy)U\cdot u=xU\cdot(yU\cdot u)$. Die Pseudo-Automorphismen bilden eine Gruppe. Diejenigen Permutationen von G, die jedem Element a von G ein Element $L(a)$ resp. $R(a)$ zuordnen vermöge $xL(a)=ax$ resp. $xR(a)=xa$, bilden in ihrer Gesamtheit eine Permutationsgruppe; sie enthält als Untergruppe diejenigen Permutationen, die das Einselement von G festlassen, die sog. inneren Abbildungen von G. Dann gilt: jede innere Abbildung von G ist ein Pseudo-Automorphismus. Ist G speciell commutativ, so ist jeder Pseudo-Automorphismus von G ein Automorphismus von G. Der Nucleus N von G besteht aus allen Elementen n von G mit $(n, G, G)=1$; Jeder Pseudo-Automorphismus von G induciert einen Automorphismus von N. Die Fixelemente der Pseudo-Automorphismen von G spielen eine ausgezeichnete Rolle. *R. Moufang* (Frankfurt am Main).

Popova, Hélène. Logarithmétiques des quasi-groupes finis. C. R. Acad. Sci. Paris 234, 1936–1937 (1952).

A quasi-integer of a non-associative algebra A means a class of equal powers ($r=s=\cdots$ if $x^r=x^s=\cdots$ for all x of A). Quasi-integers are added and multiplied thus: $x^{r+s}=x^rx^s$, $x^{rs}=(x^r)^s$. They constitute the logarithmetic L_A of A [Etherington, J. London Math. Soc. 16, 48–55 (1941); Proc. London Math. Soc. (2) 52, 241–252 (1951); these Rev. 3, 103; 12, 798]. Omitting "for all x of A" the logarithmetic of a particular element x is defined similarly. Among 14 lemmas, theorems and corollaries given without proof are the following: the logarithmetic L_Q of a finite quasigroup Q is, additively, a quasigroup which is a subdirect union of the logarithmetics of the elements of Q; the order of L_Q is a multiple of the order of each subquasigroup generated by an element of Q; if Q has no proper homomorph or proper subquasigroup, then the order of L_Q is a power of the order of Q. *I. M. H. Etherington* (Edinburgh).

Ellis, David, and Utz, Roy. Remarks on quasigroups and n-quasigroups. Publ. Math. Debrecen 2, 110–114 (1951).

The remarks on quasigroups concern (i) elementary properties of the right identity graph of a quasigroup [see Choudhury, Bull. Calcutta Math. Soc. 41, 211–219 (1949); these Rev. 11, 417], (ii) isotopy classes of quasigroups, and (iii) the operations of right and left division in a quasigroup. Apparently the authors were unaware of the work of T. Evans concerning (iii) [J. London Math. Soc. 24, 254–260 (1949); these Rev. 11, 327]. An n-quasigroup is defined for a positive integer n to be a groupoid in which, whenever $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are such that $a_i\neq a_j$ and $b_i\neq b_j$ for $i\neq j$, then there exist unique elements x and y such that $a_ix=b_i$ and $ya_i=b_i$ $(i=1, 2, \cdots, n)$. A 1-quasigroup is a quasigroup. An isotope of an n-quasigroup is an n-quasigroup. The only finite n-quasigroup, $n>1$, is the group of order 2. It is stated without proof that infinite quasigroups exist for every n. It is noted that an associative n-quasigroup is a group, and it is proved that this group is commutative if $n>1$. The authors might have noted that if Q is an n-quasigroup, $n>1$, and if (i) Q has a one-sided unit, or (ii) Q is also an m-quasigroup, $m\neq n$, then Q is finite. Thus an associative n-quasigroup, $n>1$, is a group of order 2. An n-quasigroup is descriptive of a generalized semimetric ground space in which each n distinct points form a complete metric base as defined by Ellis [Tôhoku Math. J. 3, 270–272 (1951); these Rev. 13, 970]. *J. Kiokemeister.*

Skolem, Th. Some remarks on semi-groups. Norske Vid. Selsk. Forh., Trondheim 24, 42–47 (1951).

The main results of this note are the following: a) a semi-group S in which left division is always possible is the direct product of a group G and a semi-group E in which $ab=b$ for all a, b; b) a semi-group in which the left cancellation law holds and in which any two elements have a common right-multiple has a homomorph in which both cancellation laws hold. The statement made at the beginning of the paper to the effect that a semi-group in which both cancellation laws hold can be embedded in a group is incorrect [see Malcev, Math. Ann. 113, 686–691 (1937)] but is valid in the special case considered, viz., where every pair of elements has a common right multiple [O. Ore, Ann. of Math. 32, 463–477 (1931)]. *D. Rees* (Cambridge, England).

Skolem, Th. Theorems of divisibility in some semi-groups. Norske Vid. Selsk. Forh., Trondheim 24, 48–53 (1951).

It is shown that in a commutative semi-group with cancellation in which also, 1) there exists a unity e, 2) every pair of elements has a greatest common divisor, 3) there exists no infinite sequence $a_0, a_1, \cdots$ such that a_{r+1} divides a_r but a_r does not divide a_{r+1}, the unique factorisation theorem holds. *D. Rees* (Cambridge, England).

Lesieur, Léonce. Théorèmes de décomposition dans certains demi-groupes réticulés satisfaisant à la condition de chaîne descendante affaiblie. C. R. Acad. Sci. Paris 234, 2250–2252 (1952).

The author gives sufficient conditions for the representation of an element of a complete $\cap$-continuous lattice, or partially ordered semigroup, as the intersection of a finite number of elements. *P. M. Whitman.*

Zappa, Guido. Sugli omomorfismi del reticolo dei sottogruppi di un gruppo finito. Ricerche Mat. 1, 78–106 (1952).

Suppose ϕ is a lattice homomorphism of the lattice of subgroups of a finite group G onto a lattice L. Denote by G_0 the smallest subgroup of G with $\phi(G_0)=I$, by E the largest with $\phi(E)=0$, and set $E_0=E\cap G_0$. Then it is proved that $\phi(X)=\phi_0(\chi(\psi(X)))$, where ψ is the homomorphism carrying each subgroup of G into its intersection with G_0, χ is the homomorphism carrying each subgroup of G into its

union with E_0, and ϕ_0 is ϕ restricted to subgroups contained in G_0 and containing E_0. From this is obtained a detailed necessary condition on the nature and relationship of the subgroups of G in order for G to be lattice-homomorphic to L. A similar sufficient condition is also obtained. Although these two conditions are not formally identical, it is stated that together they yield a necessary and sufficient condition.
P. M. Whitman (Silver Spring, Md.).

Herstein, I. N., and Adney, J. E. A note on the automorphism group of a finite group. Amer. Math. Monthly 59, 309–310 (1952).

If p^2 divides the order of a group G, p a prime, then G has an automorphism of order p. *K. A. Hirsch.*

Suzuki, Michio. On the finite group with a complete partition. J. Math. Soc. Japan 2, 165–185 (1950).

A group G which is the set-theoretical sum of non-overlapping subgroups H_i (apart from the unit-element) is said to admit the partition $\{H_i\}$. If the components H_i are cyclic, the partition is called complete. Groups with a partition were first studied by P. G. Kontorovič [Mat. Sbornik **5(47)**, 289–296 (1939); **7(49)**, 27–33 (1940); **12(54)**, 56–70 (1943); **19(61)**, 287–308 (1946); **22(64)**, 79–100 (1948); **26(68)**, 311–320 (1950); these Rev. **2**, 3, 4; **5**, 144; **8**, 437; **9**, 493; **11**, 579]. While this author has turned his attention more to the case of infinite groups, the paper under review deals with finite groups only. §1 redevelops some of Kontorovič's earlier results. A p-group possesses a complete partition (c.p.) if and only if it is cyclic, or all elements have order p, or it is a dihedral group. If a group with a c.p. is directly decomposable, then it is either cyclic, or a p-group with elements of order p only, or the direct product of a cyclic group of order p and a (Hölder) group with cyclic Sylow subgroups, where, moreover, one generator has also order p; a similar result holds for groups with c.p. and a non-trivial centre. §2 gives a complete survey over soluble groups with c.p. They are essentially of the types previously obtained with additional arithmetical restrictions, together with the S_4. In §3 the author shows that the property of having a c.p. is hereditary in factor-groups. (This does not hold for infinite groups: the free groups with $n>2$ generators have a c.p. [see M. Takahasi, Osaka Math. J. 1, 49–51 (1949); these Rev. 11, 7; and the last but one paper cited above], while their factor-groups, of course, need not have a c.p.). If G is a group with a c.p., neither soluble nor simple, then it possesses exactly one normal subgroup, and that is of index 2. This result leads in §4 to the further result that such a group is isomorphic to the linear fractional group of one variable over a Galois field of characteristic $\neq 2$. The structure of simple groups with a c.p. is not known.
K. A. Hirsch (London).

Higman, Donald Gordon. Focal series in finite groups. Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. ii+1+i pp.

Rédei, L. Ein Satz über die endlichen einfachen Gruppen. Acta Math. **84**, 129–153 (1950).

Finite non-Abelian groups in which all subgroups are Abelian—or even nilpotent—are well known [G. A. Miller and H. C. Moreno, Trans. Amer. Math. Soc. **4**, 398–404 (1903); O. Yu. Šmidt, Mat. Sbornik **31**, 366–372 (1924); Yu. A. Gol'fand, Doklady Akad. Nauk SSSR **60**, 1313–1315 (1948); these Rev. **9**, 565; and the author's paper cited below]. In particular, their orders are divisible by at most two primes and hence they are soluble. The author raises the problem whether a group can be simple, if in it all maximal subgroups of maximal subgroups are Abelian. This is so in the A_5 of order 60, and the author proves that for even order of the group this is the only possible case. He also proves the following theorem: A group is simple, if all its maximal subgroups are without centre (hence non-Abelian), but with Abelian maximal subgroups only. Conversely, if in a simple group the maximal subgroups of maximal subgroups are all Abelian, then the maximal subgroups are all without centre. The theorem may be empty (in view of the above remark on the A_5) if the conjecture about non-existence of simple groups of odd order is true.

The proofs are quite elementary, but rather lengthy, and require a lot of arithmetical case distinctions. They are based partly on the author's "skew product" [Comment. Math. Helv. **20**, 225–264 (1947); these Rev. **9**, 131]. The author also gives a very complete list of properties which his hypothetical simple groups would possess.
K. A. Hirsch (London).

Rédei, L. Die endlichen Gruppen ohne direkt unzerlegbare Untergruppen. Math. Ann. **122**, 127–130 (1950).

The author determines all finite groups which do not possess any directly decomposable subgroups. He calls them α-groups. The only Abelian α-groups are obviously the cyclic p-groups. The only non-Abelian α-p-groups are the generalized quaternion-groups Q_f of order 2^{f+1}, generated by two elements B, C with the relations $B^{2^f}=1$, $C^2=B^{2^{f-1}}$, $CBC^{-1}=B^{-1}$, $f\geqq 2$. The remaining α-groups turn out to be exhausted by the following two types:

I. $A^{p^e}=B^{q^f}=1,\quad BAB^{-1}=A^a;$
$\quad q^f|p-1;\quad e, f\geqq 1;\quad p^e|a^{q^f}-1,\quad p\nmid a^{q^{f-1}}-1;$

II. $A^{p^e}=B^{2^f},\quad C^2=B^{2^{f-1}},\quad CBC^{-1}=B^{-1},$
$\qquad BAB^{-1}=A^a,\quad CAC^{-1}=A^{a^{2^{f-1}}};$
$\quad 2^f|p-1;\quad e\geqq 1,\quad f\geqq 2;\quad p^e|a^{2^{f-1}}+1.$

The proof is based on the remark that the Sylow-subgroups of an α-group must be α-groups and hence cyclic or of type Q_f; groups with cyclic Sylow-subgroups are, of course, known since Hölder's days. *K. A. Hirsch* (London).

Rédei, L. Über die Basen endlicher Gruppen. Math. Z. **53**, 454–455 (1951).

Let $G=\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ be a finite group $\neq 1$. The author calls the basis $\alpha_1, \alpha_2, \cdots, \alpha_k$ of G (i) primary, if the α_i are of prime-power order; (ii) minimal, if in addition no α_i is superfluous; (iii) reduced, if in addition no α_i can be replaced by a power of α_i of smaller order. The author shows that in p-groups all minimal bases are reduced. (Simplest case: $G=\{\alpha, \beta\}$, a non-cyclic p-group. Then $\{\alpha^p, \beta\}$ is a proper subgroup.) He proves the following more general result. Theorem. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be a minimal basis of the finite nilpotent group G, p_i the prime divisor of the order of α_i. Let $\mathfrak{m}$ be any set of indices taken from $1, 2, \cdots, k$ and put $G_{\mathfrak{m}}=\{\beta_1, \beta_2, \cdots, \beta_k\}$, where β_i equals $\alpha_i^{p_i}$ or α_i according as $i\in\mathfrak{m}$ or not. Then the 2^k groups $G_{\mathfrak{m}}$ are all distinct from one another. *K. A. Hirsch* (London).

Piccard, Sophie. Les permutations associées aux bases du groupe de Klein généralisé et les groupes associés. C. R. Acad. Sci. Paris **233**, 906–908 (1951).

The generalized Klein group G_{2^n} is a regular permutation group of degree and order 2^n, generated by systems of n independent elements called bases. (Each element of one

base could be described as the permutation of the 2^n n-digit binary numbers 0 to 2^n-1 effected by changing a fixed binary digit in each). If $a_1\cdots a_{2^n}$ is a permutation of $1\cdots 2^n$ such that the permutations defined by changing a fixed binary digit in the subscripts form the same basis, then this is one of $2^n n!$ permutations "associated with the basis", forming an associated group g. The author studies these associated groups, and the group of permutations commutative with each element of G_{2^n}. *J. S. Frame.*

Checcucci, Vittorio. I gruppi abeliani di omografie dello spazio ordinario. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 229–264 (1951).

Using the methods of S. Cherubino [same Rend. 9, 177–188 (1950); these Rev. 13, 104] the author classifies the abelian subgroups of the projective group in complex 3-space. A combination of algebraic and geometrical techniques is used to show that these groups fall into some seventeen types. *S. A. Jennings.* (Vancouver, B. C.).

Mišina, A. P. On complete direct sums of Abelian groups of the first rank without torsion. Ukrain. Mat. Žurnal 2, no. 4, 64–70 (1950). (Russian)

A group with the decomposition described in the title is completely decomposable (that is, representable as an ordinary direct sum of groups of the first rank) if and only if the number of summands (of the complete direct sum) which are distinct from the additive group of all rational numbers is finite. *R. A. Good* (College Park, Md.).

Gluškov, V. M. On some questions of the theory of nilpotent and locally nilpotent groups without torsion. Mat. Sbornik N.S. 30(72), 79–104 (1952). (Russian)

Throughout this review $\mathfrak{G}$ denotes a locally nilpotent torsion-free group. The group $\mathfrak{G}$ is nilpotent with finite special rank k if and only if it has a rational series of length k. In $\mathfrak{G}$ finitely many subgroups with finite special rank generate a nilpotent subgroup with finite special rank. A group is called strongly locally nilpotent whenever every finite set of its nilpotent subgroups with finite special rank generates a nilpotent group. Not every locally nilpotent group is strongly so. The composite of arbitrarily many complete (complete in the sense of unrestricted root extraction) subgroups of $\mathfrak{G}$ is complete; such a result is invalid for periodic groups. If $\mathfrak{G}$ is complete, every term of the lower central series is a complete group. A group $\mathfrak{G}$ with the minimal condition for servant normal subgroups is called an M_0-group. A ZA-group in which the factors of the upper central series have finite special rank is called a ZAF-group. An M_0-group is a (torsion-free) ZAF-group; the converse is an open question. An M_0-group is solvable. Each of the two conditions is necessary and sufficient that a torsion-free group be nilpotent with finite special rank: it is locally nilpotent with the minimal condition for servant subgroups; or it is locally nilpotent with the maximal condition for servant normal subgroups. Let a torsion-free group $\mathfrak{H}$ be the extension either of a ZA-group by means of a group with finite special rank or else of a group satisfying the minimal condition for servant normal subgroups by means of a ZA-group; then $\mathfrak{H}$ is a ZA-group itself if and only if it is locally nilpotent. Another theorem describes a group $\mathfrak{G}$ which is the extension of an abelian normal subgroup by means of a group with finite special rank. Triangular matrices with order infinity (but not necessarily order ω) are used to construct an example of an M_0-group which is not the extension of an abelian group by means of a group with finite special rank. Each of the two conditions is necessary and sufficient that an M_0-group have such a structure: one of its maximal abelian normal subgroups is contained in its hypercenter with index not exceeding the first limiting ordinal; or its upper central series has length less than the second limiting ordinal. In a torsion-free ZA-group if one of its maximal abelian normal subgroups has finite special rank n, then the special rank of the group is finite and does not exceed $\frac{1}{2}n(n+1)$. A class of groups of matrices is cited, each member of which is a locally nilpotent group coinciding with its commutant. *R. A. Good* (College Park, Md.).

Plotkin, B. I. On the theory of noncommutative groups without torsion. Mat. Sbornik N.S. 30(72), 197–212 (1952). (Russian)

Several properties concerning the isolator $I(H)$ of a subgroup H of the group $\mathfrak{G}$ are stated as lemmas. A subgroup of an R-group and its isolator have the same centralizer. If every subgroup of the R-group $\mathfrak{G}$ has periodic index with respect to its isolator, then $Z_\alpha(I(H))=I(Z_\alpha(H))$, where H is any subgroup of $\mathfrak{G}$ and $Z_\alpha(M)$ denotes the αth hypercenter of the subgroup M. If $\mathfrak{G}$ is an R-group containing a subgroup H with an ascending central series, then $I(H)$ also has an ascending central series, and the classes of H and $I(H)$ are equal. A torsion-free abelian group of rank one is called rational. Every dense subgroup of a group with rational series is rational. An R-group $\mathfrak{G}$ with an ascending invariant rational series contains a normal subgroup $\mathfrak{G}'$ with an ascending central series such that $\mathfrak{G}/\mathfrak{G}'$ is a torsion-free abelian group and every rational component in a decomposition of the group $\mathfrak{G}$ which lies in $\mathfrak{G}'$ is cyclic. If an arbitrary group $\mathfrak{G}$ has a rational series of finite length n, then every rational series for $\mathfrak{G}$ has length n, so that n may be called the rational rank of $\mathfrak{G}$ and denoted by $r(\mathfrak{G})$. In such case the length of a rational series for a proper isolated subgroup of $\mathfrak{G}$ is less than n. Conditions under which $r(F)=r(I(F))$ for certain subgroups F of $\mathfrak{G}$ are given. A torsion-free group $\mathfrak{G}$ is said to satisfy condition (N) provided, for subgroups A and B of $\mathfrak{G}$, whenever A is a maximal isolated subgroup of B, then A is normal in B. An R-group satisfying (N) and having a finite rational series is nilpotent. If an R-group $\mathfrak{G}$ satisfies (N), then an invariant ascending rational series of $\mathfrak{G}$ is a central series. *R. A. Good* (College Park, Md.).

Haimo, Franklin. Groups with a certain condition on conjugates. Canadian J. Math. 4, 369–372 (1952).

If there is a positive integer α such that $x^\alpha=1$ for every element x of the group G, then G is said to have uniform torsion [u.t.], and the minimum such α is called the exponent of G. Denote by $Z^{(i)}$ the ith term of the upper central series of the group G, $Z^{(1)}=$ the center of G. If the number of conjugates to each element of G does not exceed the positive integer M, then $G/Z^{(1)}$ has u.t. with exponent a divisor of $M!$. Now assume that $G/Z^{(1)}$ has u.t. with exponent α. Then for N a normal subgroup of G, $[G, N]/[G, [G, N]]$ has u.t. with exponent dividing α; $[G, Z^{(i+1)}]$ has u.t. with exponent $\alpha(i)$ dividing α^i and $\alpha(i+1)$. Furthermore, if $\gamma(i)=\alpha\cdot\alpha(i-1)$, the mapping $x\to x^{\gamma(i)}$ is an endomorphism of $Z^{(i)}$ into $Z^{(1)}$. Some consequences of this last result are noted; for instance, if G is the set-theoretical sum of the subgroups $Z^{(i)}$ $[i=1, 2, \cdots]$, and if $G/Z^{(1)}$ has u.t., then $[G, G]$ is a torsion group [i.e. is without elements of order 0]. If each element of the ith term ${}^iG$ of the lower central series of G is contained in some term of the upper central series, and if $G/Z^{(1)}$ is u.t., then ${}^kG$ is a torsion group for all $k>i$, and the jth derived group $G^{(j)}$ is a torsion group for

sufficiently large j. For related results, see R. Baer [Math. Ann. 124, 161–177 (1952); these Rev. 13, 622] and B. H. Neumann [Proc. London Math. Soc. (3) 1, 178–187 (1951); these Rev. 13, 316]. *D. Higman* (Urbana, Ill.).

Ayoub, Christine Williams. A theory of normal chains. Canadian J. Math. 4, 162–188 (1952).

An M-ϕ group G is a group with an operator system M and a complete lattice ϕ of admissible subgroups (called ϕ subgroups). A ϕ subgroup of G is again an M-ϕ group, and so is a factor group of G by a normal ϕ subgroup. Homomorphism, etc. between M-ϕ groups is defined in the natural way, and the analogues of the classical theorems hold.

Let (K) be a property of ϕ subgroups of an M-ϕ group. A chain

$$(1)\qquad 1=N_0\subseteq N_1\subseteq\cdots\subseteq N_i\subseteq N_{i+1}\subseteq\cdots$$

of subgroups of an M-ϕ group is an ascending K-chain if, for all i, N_i is a normal ϕ subgroup of G, and N_{i+1}/N_i satisfies (K) in G/N_i. The upper K-chain of G, when it exists, is the K-chain

$$(2)\qquad 1=T_0\subseteq T_1\subseteq\cdots\subseteq T_i\subseteq T_{i+1}\subseteq\cdots$$

such that, for all i, and for all K-chains (1), $N_i\subseteq T_i$. Descending K-chains, and the lower K-chain, are defined similarly. The author obtains conditions sufficient for the existence of upper and lower K-chains, and discusses their properties.

The discussion is applied to three particular cases. Loewy chains are obtained by taking (K) to be: A satisfies (K) in G if it is contained in the join of minimal normal ϕ subgroups of G. The upper Loewy chain always exists; the lower Loewy chain exists if the normal ϕ subgroups satisfy the minimal condition. Central chains are obtained by taking (K) to be: A satisfies (K) in G if it is in the centre of G. Both upper and lower central chains always exist. (This is not quite trivial, since the centre of G is not necessarily a ϕ subgroup.) If there is a finite Loewy chain connecting 1 to G, then the upper and lower central chains are also finite. If G is ϕ nilpotent of finite class (i.e. if there is a finite central chain connecting 1 to G) then any Loewy chain is a central chain. Finally, S-chains are obtained by taking (K) to be: A satisfies (K) in G if it is ϕ nilpotent of finite class. The upper and lower S-chains exist provided that there exists a Loewy chain connecting 1 to G, and that ϕ is normal (i.e. $S\epsilon\phi$ implies $g^{-1}Sg\epsilon\phi$). G is said to be ϕ soluble if it possesses a finite S-chain connecting 1 to G. It is shown that this coincides with the more normal definition, and that if G is ϕ soluble, the factors of any Loewy chain connecting 1 to G are abelian. *G. Higman* (Manchester).

Ayoub, Christine Williams. On the primary subgroups of a group. Trans. Amer. Math. Soc. 72, 450–466 (1952).

An M-ϕ group [see the preceding review] is called primary of characteristic F if it has a composition series all of whose factor groups are M-ϕ isomorphic to F. This paper considers conditions under which an M-ϕ group is a direct product of primary subgroups. Assume throughout that G is an M-ϕ group with a composition series, in which inner automorphisms are M-ϕ automorphisms.

Following Wielandt [Math. Z. 45, 209–244 (1939)] a subgroup S of G is called a ϕ link if there is a normal ϕ chain connecting S to G. The ϕ links form a sublattice of the lattice of ϕ subgroups; and if A, B are ϕ links, the composition factors from $A\cap B$ to B are, except for multiplicity, the same as those from A to $\{A, B\}$. Thus if A, B are primary with the same characteristic, $\{A, B\}$ is also primary. On the other hand, primary ϕ links with different characteristics generate their direct product. Now call an M-ϕ group unitoral if it has a unique maximal normal ϕ subgroup. Then G is the join of a finite number of unitoral ϕ links, and if G is a direct product of primary groups its unitoral ϕ links are contained in the direct factors. Thus a necessary and sufficient condition that G is a direct product of primary groups is that all unitoral ϕ links of G are primary.

Next the author turns to generalisations of Burnside's theorem that a finite group is nilpotent if and only if it is a direct product of primary groups with abelian characteristics. Difficulties arise because in general a primary M-ϕ group with abelian characteristic need not be ϕ nilpotent, as is shown by the example of a non-abelian 2-parameter Lie group, the ϕ subgroups being the closed connected subgroups. However, if it is assumed that ϕ cyclic ϕ subgroups are abelian, and either that every element has finite order, or that every ϕ subgroup has a finite number of conjugates, then a direct product of primary groups of abelian characteristic is ϕ nilpotent. The converse implication requires the assumption that unitoral ϕ cyclic ϕ subgroups are primary. (It should be noted that in this paper "ϕ nilpotent" means "ϕ nilpotent of finite class", whereas in the paper of the preceding review it is given a much wider sense.) *G. Higman* (Manchester).

Itô, Noboru. Note on A-groups. Nagoya Math. J. 4, 79–81 (1952).

An A-group is a soluble group whose Sylow subgroups are all abelian. The author shows that the irreducible representations of an A-group are similar to monomial forms. Let further N denote the largest nilpotent normal subgroup of an A-group $\mathfrak{G}$, and $X(\mathfrak{G})$ the set of all elements G of $\mathfrak{G}$ such that $\chi(G)\neq 0$ for any simple character χ of $\mathfrak{G}$. By applying a previous result on characters of defect zero [Nagoya Math. J. 3, 31–48 (1951); these Rev. 13, 431; there is a misquotation of a theorem in this review; the correct form of the theorem in question is: If a soluble group G has no normal p-subgroups and no subgroups which have as a homomorphic image a group of the first kind, then G has a character of defect zero], the author shows that $\{X(\mathfrak{G})\}=N$. If the order of $\mathfrak{G}$ is divisible only by two distinct primes then the sharper result $X(\mathfrak{G})=N$ is valid. *J. Levitzki.*

Suzuki, Michio. A characterization of simple groups $LF(2, p)$. J. Fac. Sci. Univ. Tokyo. Sect. I. 6, 259–293 (1951).

As usual, we denote by $LF(2, p)$ the simple group of order $\frac{1}{2}p(p-1)(p+1)$, ($p>3$ a prime number). All proper subgroups of $LF(2, p)$ are of one of the following four types: metacyclic groups (i.e. groups with a cyclic normal subgroup such that the factor group is cyclic), tetrahedral, octahedral, icosahedral groups. The author proves the converse. If all proper subgroups of a non-cyclic simple group G are of one of these four types, then G is isomorphic to a group $LF(2, p)$. More generally, if G is a non-cyclic simple group and if all proper subgroups of G are metacyclic groups, symmetric groups S_n, alternating groups A_n, semisimple groups, then G is again isomorphic to a group $LF(2, p)$. Here, by a semisimple group, we mean a group which does not possess a soluble normal subgroup $\neq 1$. The proof is ingenious but long and difficult. *R. Brauer* (Cambridge, Mass.).

Schuff, Hans Konrad. Über Wurzeln von Gruppenpolynomen. Math. Ann. 124, 294–297 (1952).

A group-polynomial $f(x_1, \cdots, x_n)$ over a group G is an element of the free product F of G and the free group with $x_1, \cdots, x_n$ as free generators. If $h_1, \cdots, h_n$ belong to a group H containing G, then $f(h_1, \cdots, h_n)$ is the image of $f(x_1, \cdots, x_n)$ under the homomorphism of F into H which is the identity on G and maps x_i onto h_i, $i=1, \cdots, n$. It is proved that if (i) the weight of f in some x_i is non-zero and (ii) $f(1, \cdots, 1)$ is in the centre of G, then the equation $f(h_1, \cdots, h_n)=1$ has a solution in some H containing G.

G. Higman (Manchester).

Scott, W. R. Means in groups. Amer. J. Math. 74, 667–675 (1952).

Schimmack [Math. Ann. 68, 125–132 (1909)] gave certain postulates for a sequence of mean value functions $f_n=f_n(x_1, \cdots, x_n)$, $n=1, 2, \cdots$, on a group G, and showed that if G is the additive group R of reals then f_n is the arithmetic mean $(x_1+\cdots+x_n)/n$. Beetle [ibid. 76, 444–446 (1915)] showed that if $G=R$ then Schimmack's postulates are completely independent. The author now shows that the conclusions of Schimmack and Beetle are valid for any infinite Abelian torsion-free group G such that $mG=G$ for all positive integers m.

E. F. Beckenbach.

Maak, Wilhelm. Integralmittelwerte von Funktionen auf Gruppen und Halbgruppen. J. Reine Angew. Math. 190, 34–48 (1952).

The author investigates, with a view to applications in ergodic theory, various classes of functions on groups, or more generally on semi-groups, for which it is possible to define a mean value in generalization of the procedure previously used by the author for proving the existence of a mean value for an almost periodic (a.p.) function on a group [Abh. Math. Sem. Univ. Hamburg 11, 240–244 (1935)]. For some of the classes there are indicated applications and further results the proof of which will be published elsewhere, thus for instance for the class of a.p. functions on a semi-group introduced by the author in the paper reviewed below, for the class of W-a.p. functions on a group introduced by the reviewer [Mat. Tidsskr. B. 1946, 153–162; these Rev. 8, 14], and for the more narrow class of ω-a.p. functions defined in the present paper. The theory of this latter class has applications to Godement's theory of positive definite functions on groups [Trans. Amer. Math. Soc. 63, 1–84 (1948); these Rev. 9, 327] and connects with Eberlein's weakly a.p. functions on abelian groups [Trans. Amer. Math. Soc. 67, 217–240 (1949); these Rev. 12, 112].

E. Følner (Copenhagen).

Maak, Wilhelm. Fastperiodische Funktionen auf Halbgruppen. Acta Math. 87, 33–58 (1952).

Von Neumann [Trans. Amer. Math. Soc. 36, 445–492 (1934)] has generalized Bohr's almost periodic (a.p.) functions of a real variable to groups. In the present paper a generalization to semi-groups is obtained. For the additive semi-group of positive numbers Bohr previously had occasion to introduce the notion of (continuous) almost periodic function [J. Reine Angew. Math. 157, 61–65 (1926), in particular pp. 62–63]. Bohr showed that these functions extend themselves in a unique fashion to usual a.p. functions of an unrestricted real variable. The main theorems of von Neumann's theory hold in verbally the same formulation for the a.p. functions on semi-groups. If the semi-group is a group, the a.p. functions on it coincide with von Neumann's a.p. functions. Denoting by S a semi-group with unit element, the author calls a function $f(x)$ on S an a.p. function if to every $\epsilon>0$ there exist a finite number of subsets $A_1, \cdots, A_n$ of S the union of which is S and with the property that if, for some elements x, y, c', d' and suitable i, $c'xd', c'yd' \in A_i$, then $|f(cxd)-f(cyd)| \leqq \epsilon$ for all $c, d \in S$. A main result of the theory is the following approximation theorem: Every a.p. function on S can be uniformly approximated by "trigonometric polynomials", i.e. sums $\sum_{i=1}^{n} \sum_{\rho,\sigma=1}^{s_i} \gamma_{i,\rho\sigma} D_{i,\rho\sigma}(x)$ where the $D_{i,\rho\sigma}(x)$ are coefficients in unitary representations of S. In rough outlines the method of proof is easily stated. A semi-group $\hat{S}$ is obtained from S by identifying any two elements a and b for which $f(a)=f(b)$ for all a.p. functions f on S. This semi-group $\hat{S}$ can be imbedded in a group $\hat{G}$, and the a.p. functions on $\hat{S}$ extend themselves in a unique fashion to a.p. functions on $\hat{S}$. Application of von Neumann's approximation theorem to $\hat{G}$ yields the desired result for $\hat{S}$ and hence for S. Stated a little more explicitly: It is first shown that the transformations $T_af(x)=f(xa)$ are linear one-to-one transformations of the space of a.p. functions on S onto the space itself. They form a transformation semi-group which realizes the abovementioned $\hat{S}$. The transformation group which they generate is the desired extension of the semi-group $\hat{S}$ to a group $\hat{G}$. The extension $\hat{f}$ of the a.p. function f on S from $\hat{S}$ to $\hat{G}$ is given by $\hat{f}(T)=Tf(1)$. A main tool in the proofs is a combinatorial lemma previously introduced in the theory of a.p. functions on groups by the author [Abh. Math. Sem. Univ. Hamburg 11, 240–244 (1935)] (and later interpreted by Weyl as "marriage problem" [Amer. J. Math. 71, 178–205 (1949); these Rev. 10, 461, 856]). In another paper [see the preceding review] the author has used the combinatorial lemma to introduce a mean value for a.p. functions on semi-groups. Once this mean value theory has been established the road to a theory of Fourier series lies open. An a.p. function f on S and the corresponding a.p. function $\hat{f}$ on $\hat{G}$ have the same mean values and, more generally, the Fourier series of the extended function $\hat{f}$ is the termwise extension of the Fourier series of f. Thus, in particular, the Parseval equation is valid.

E. Følner (Copenhagen).

Duncan, D. G. Note on a formula by Todd. J. London Math. Soc. 27, 235–236 (1952).

For Littlewood's [The theory of group characters and matrix representation of groups, 2nd ed., Oxford, 1950; for a review of the 1st ed. see these Rev. 2, 3] multiplication of Schur functions in the representation theory of full linear groups (or of symmetric groups), Todd [Proc. Cambridge Philos. Soc. 45, 328–334 (1949); these Rev. 10, 672] has given the formula $\{\mu\} \otimes S_n = \sum_\sigma \theta_\sigma a_\mu{}^\sigma \{\sigma\}$, where μ is a partition of m, σ ranges over all partitions of mn, θ_σ is ± 1 or 0, and $a_\mu{}^\sigma$ are positive integers or 0. The present paper proves that $\theta_\sigma=0$ if the Young diagram associated with σ has a non-zero n-core; otherwise θ_σ is $+1$ or -1 according as the sum of the heights of the removed n-hooks is even or odd, in the sense of terms defined by the reviewer [Jap. J. Math. 17, 165–184 (1940); these Rev. 3, 195; 4, 340]. The proof combines Todd's observations with the recursion formula of Murnaghan and the reviewer [loc. cit.] and Robinson's [Amer. J. Math. 70, 277–294 (1948); these Rev. 10, 678] theorem on the unique determination of the alluded sum of the heights.

T. Nakayama (Nagoya).

Osima, Masaru. On the irreducible representations of the symmetric group. Canadian J. Math. 4, 381–384 (1952).

Let $T=[\alpha_i]$ be the Young diagram with n nodes belonging to the partition $n=\alpha_1+\cdots+\alpha_k$, $\alpha_1\geqq\alpha_2\geqq\cdots\geqq\alpha_k$. Let p be a prime, let T_0 be a p-core of T, let $T=\{T_0;D_1,\cdots,D_p\}$ where $D_1,\cdots,D_p$ is a p-diagram [Nakayama and Osima, Nagoya Math. J. 2, 111–117 (1951); these Rev. 12, 672]. Let $\bar{T}$ denote the associated diagram of T (i.e. T with rows and columns interchanged). If $T=\bar{T}$ then T is said to be self-associated. Two diagrams T and T' are associated if and only if $\bar{T}_0=T_0$ and $\bar{D}_i=D'_{p-i+1}$ $(i=1,\cdots,p)$ and, in particular, T is self-associated if and only if T_0 is self-associated and $\bar{D}_i=D_{p-i+1}$ $(i=1,\cdots,p)$. Let T_0 be a self-associated p-core with m nodes and let $n=m+lp$. Then the number of self-associated irreducible representations of S_n belonging to the block $B(T_0)$ is determined by l. These results are applied for the prime $p=2$ to obtain recursion formulas for the number of partitions of n and the number of self-associated diagrams with n nodes.

R. M. Thrall (Ann Arbor, Mich.).

Gel'fand, I. M., and Šapiro, Z. Ya. Representations of the group of rotations in three-dimensional space and their applications. Uspehi Matem. Nauk (N.S.) 7, no. 1(47), 3–117 (1952). (Russian)

This is a clear, full, and elementary exposition of the representations of the 3-dimensional rotation group and their applications, together with some new material. Particular stress is laid on relations with quantum mechanics and with other parts of mathematics. However, the approach is explicit, computational, and practical, and invariant formulations and questions of general theory are kept correspondingly in the background. On the whole the treatment is distinctly more detailed and comprehensive than any in English and should be particularly valuable for workers in quantum mechanics and for those interested in a highly concrete introduction to the theory of representations of Lie groups. In addition to the usual material, including spherical harmonics, decomposition of product representations, spinors, and tensor representations, there are three sections containing some new material. These consist of: 1) explicit determination of the matrix elements of all the irreducible representations; 2) a study of the decomposition of vector and tensor fields under the action of the rotation group, with application to Maxwell's equations; 3) a study of equations invariant under the rotation group and of the Dirac equation in particular. *I. E. Segal.*

Orihara, Masae, and Tsuda, Takeo. The two sided regular representation of a locally compact group. Mem. Fac. Sci. Kyūsyū Univ. A. 6, 21–29 (1951).

This paper treats the theorem that if G is a locally compact group and L and R the rings of operators on $L_2(G)$ generated respectively by left and right translations, then $L'=R$, i.e., every bounded linear operator on $L_2(G)$ which commutes with every element of L is in R. The same theorem has been treated in a more abstract setting by Dixmier [C. R. Acad. Sci. Paris 233, 837–839 (1951); these Rev. 13, 472]. The theorem had previously been proved for the case in which G is unimodular [cf., e.g., Segal, Ann. of Math. (2) 51, 293–298 (1950); these Rev. 12, 157] and the present treatment is along similar lines, but is incomplete in that unjustified use of the Fubini theorem is made in the proof of Lemma 5.

I. E. Segal (Chicago, Ill.).

Yamabe, Hidehiko. On a problem of Chevalley. Sûgaku 4, 17–21 (1952). (Japanese)

Let G be a finite-dimensional, locally compact, locally connected topological group, containing no arbitrarily small subgroup. It is known that such a group G has a neighborhood of e which is covered by a family F of one-parameter local semigroups. The author proves the existence of a cross-section C to the family F, using a Haar measure on G. Let, then, $\{U_i\}$ be a finite open covering of C and let L_i be the intersection of all closed local semigroups, containing every one-parameter semigroup which connects e with some point in the intersection of U_i and the interior of C. The author proves that G is a Lie group under the following assumption: for sufficiently small U_i, there exists a neighborhood V of e such that the intersections of $V\cap L_i$ and $(V\cap L_i)^{-1}$ are e. For the proof the author uses a Haar measure of G to define a well-behaved function $\phi(x,y)$ on $G\times G$ and applies Kuranishi's theorem on the extension of Lie groups [cf. Proc. Amer. Math. Soc. 1, 372–380 (1950); these Rev. 12, 77]. *K. Iwasawa* (Cambridge, Mass.).

Kuranishi, Masatake. Developments in the theory of topological groups. Sûgaku 4, 40–49 (1952). (Japanese)

This is a report on recent developments in the theory of locally compact topological groups, dealing mainly with the following papers: A. Gleason, Proc. Nat. Acad. Sci. U. S. A. 36, 663–667 (1950) [these Rev. 12, 391]; D. Montgomery, Ann. of Math. 48, 650–658 (1947); 49, 110–117 (1948); 50, 570–580 (1949); 52, 261–271, 591–605 (1950) [these Rev. 9, 174, 496; 11, 10; 13, 319; 12, 673]; D. Montgomery and L. Zippin, the paper reviewed below. *K. Iwasawa.*

Montgomery, Deane, and Zippin, Leo. Existence of subgroups isomorphic to the real numbers. Ann. of Math. (2) 53, 298–326 (1951).

By means of an ingenious but rather tortuous argument the authors prove that every connected finite-dimensional locally compact metrizable group which is not compact contains a closed subgroup isomorphic to the real numbers. Lemma 5.2 omits the hypothesis $x\epsilon K$, which, however, is satisfied in subsequent references to the lemma.

A. Gleason (Cambridge, Mass.).

Montgomery, Deane, and Zippin, Leo. Two-dimensional subgroups. Proc. Amer. Math. Soc. 2, 822–838 (1951).

It is proved that every separable metric, connected, locally compact, noncompact, n-dimensional group G with $n\geqq 2$ has a noncompact, connected, two-dimensional subgroup. Obviously this follows immediately from the theorem that every noncompact, connected n-dimensional group G, $n>2$, has a noncompact connected subgroup of dimension greater than one and less than n. An outline of the proof of this theorem is as follows. It is easily reduced to the case where G has no center and no one-dimensional normal subgroup. In that case, let L be an n-dimensional, connected, locally connected, locally compact group which is mapped onto an everywhere dense subgroup of G by a one-to-one continuous homomorphism, and let Q be a subgroup of L which is isomorphic to the reals [cf. the paper reviewed above]. Then, under the assumption that the above theorem is false, it is proved that the set of conjugates of the positive part Q^+ of Q fills out L without overlap except at e. These transforms have a cross section S with at least some of the homotopy properties of an $(n-1)$-sphere. Hence S can have no nontrivial covering because $n-1\geqq 2$. But on the other

hand it is shown that S does have a covering. In this way the assumption that the theorem is false leads to a contradiction. *K. Iwasawa* (Cambridge, Mass.).

√**★Montgomery, Deane. Properties of finite-dimensional groups.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 442–446. Amer. Math. Soc., Providence, R. I., 1952.

The author first reviews our meager knowledge and points out the many unsolved problems concerning the action of finite-dimensional locally compact groups on finite-dimensional spaces. Next he discusses the structure of the groups themselves. The paper concludes with two theorems by Montgomery and Zippin which were new at the time of the International Congress but which have since been published [see the two preceding reviews]. *A. M. Gleason.*

Montgomery, Deane, and Zippin, Leo. Four-dimensional groups. Ann. of Math. (2) 55, 140–166 (1952).

The principal result of this paper is the theorem that four-dimensional separable metric locally compact, connected and locally connected groups are Lie groups, i.e. the solution of the Hilbert's fifth problem for the case of dimension four. The outline of the proof is as follows. Let G be a topological group with the properties stated above and let G be also non-compact. Using the existence theorem of non-compact two-dimensional subgroups in such a group G [cf. the second preceding review], the authors first show that G is locally euclidean and contains no arbitrarily small subgroup. It then follows that, if there exists in G a non-trivial proper closed normal subgroup N, then N and G/N are both Lie groups of dimension one, two or three and, consequently, that G itself is also a Lie group. The rest of the paper is therefore devoted to the proof of the existence of such a normal subgroup N and this is done by a subtle topological method, using again two-dimensional subgroups in G and a cross-section of local one-parameter subgroups which cover a neighborhood of the identity of G. *K. Iwasawa.*

Wang, Hsien-Chung. One-dimensional cohomology group of locally compact metrically homogeneous spaces. Duke Math. J. 19, 303–310 (1952).

The author studies metric spaces X which are metrically homogeneous (the group of isometries acts transitively on X). If such an X is compact, it is the projective limit of homogeneous spaces of compact Lie groups, and this situation has been studied extensively. In the present paper some results are obtained for a space X which is connected, locally connected, locally compact, non-compact, separable, and metrically homogeneous. By showing that such a space is the homogeneous space of a connected, locally compact, separable, metric group G by a compact subgroup H, it is proved that X is contractible in its one point compactification X'. Using this result it is shown that if w: $X\to X'$ denotes the inclusion map then the induced homomorphism $w^{\#}: H^1(X')\to H^1(X)$ is trivial. Using the fact that the kernel of $w^{\#}$ is isomorphic to the augmented group $H^0(E)$ of the space of ends of X (this is a generalization of a result of Specker [Comment. Math. Helv. 23, 303–333 (1949); these Rev. 11, 451]) and the fact that X can have at most two ends, it follows that $H^1(X')$ is either trivial or isomorphic to the coefficient group. In the latter case, X is shown to be homeomorphic to the product of the real line and a compact space. *E. H. Spanier* (Chicago, Ill.).

Harada, Shigeharu. Remarks on the topological group of measure preserving transformations. Proc. Japan Acad. 27, 523–526 (1951).

Let G be the group of all measure-preserving transformations of the closed unit interval onto itself, topologized by the neighborhood topology [cf. Halmos, Trans. Amer. Math. Soc. 55, 1–18 (1944); these Rev. 5, 189]. Using the results of the reviewer on conjugate classes in G [Ann. of Math. (2) 45, 786–792 (1944); these Rev. 6, 131], the author proves that G is (1) (topologically) simple and (2) arcwise connected. *P. R. Halmos* (Chicago, Ill.).

Clifford, A. H. A class of partially ordered abelian groups related to Ky Fan's characterizing subgroups. Amer. J. Math. 74, 347–356 (1952).

Dans un groupe ordonné G, un élément $f\geqq 0$ est dit archimédien si pour tout $g\epsilon G$ il existe un entier $n>0$ tel que $nf\geqq g$. Un sous-groupe H de G est dit singulier s'il est filtrant et ne contient aucun élément archimédien; un sous-groupe H de G est dit convexe s'il est distinct de G et si les relations $h_1\epsilon H$, $h_2\epsilon H$, $h_1\geqq f\geqq h_2$, entraînent $f\epsilon H$. Généralisant des résultats de Ky Fan [Ann. of Math. 51, 409–427 (1950); ces Rev. 11, 525], l'auteur considère les groupes G de fonctions numériques définies dans un ensemble S et satisfaisant aux conditions suivantes: P_1) G contient les fonctions constantes; P_2) pour tout couple de points distincts x_1, x_2 de S, il existe $f\epsilon G$ telle que $f(x_1)\neq f(x_2)$; P_3) si $f(x_0)=0$, où $f\epsilon G$, il existe $g\geqq 0$ dans G telle que $g\geqq f$ et $g(x_0)=0$; P_4) si $f\epsilon G$ est non-archimédien, alors il existe $x_0\epsilon S$ tel que $f(x_0)=0$; P_5) si ϕ est un ensemble de fonctions $\geqq 0$ de G tel que tout sous-ensemble fini de fonctions de ϕ a un zéro commun dans S, alors toutes les fonctions de ϕ ont un zéro commun dans S. Il prouve que ces groupes G peuvent être caractérisés abstraitement par les quatre conditions suivantes: I) G contient un sous-groupe R isomorphe au groupe ordonné des réels; II) Un élément au moins de R est archimédien dans G; III) Si $nf+g\geqq 0$ pour tout entier $n>0$, alors $f\geqq 0$; IV') Tout sous-groupe singulier maximal de G est un sous-groupe convexe maximal. En outre, l'ensemble S est alors en correspondance canonique avec l'ensemble des sous-groupes singuliers maximaux de G. Lorsqu'on munit S de la topologie la moins fine rendant continues les fonctions de G, S est complètement régulier, mais non nécessairement compact; l'auteur montre qu'une condition IV) donnée par Ky Fan (loc. cit.) pour que S soit compact n'est pas nécessaire. *J. Dieudonné* (Ann Arbor, Mich.).

NUMBER THEORY

Goodman, A. W., and Zaring, W. M. Euclid's algorithm and the least-remainder algorithm. Amer. Math. Monthly 59, 156–159 (1952).

The least-remainder algorithm for finding the g.c.d. of two positive integers b_1 and b_2 ($b_1>b_2$) consists of the m equations

$$b_i=b_{i+1}q_{i+2}+\epsilon_{i+2}b_{i+2}\quad(i=1,\cdots,m-1),\qquad b_m=b_{m+1}q_{m+2},$$

where the ϵ's are ±1, and such that $b_i\neq 0$ ($i=1,\cdots,m+2$),

$$-\tfrac{1}{2}b_{i-1}<\epsilon_ib_i\leqq\tfrac{1}{2}b_{i-1}\quad(i=3,\cdots,m+2).$$

The authors show that the number of equations in Euclid's algorithm exceeds this number m by just the number of negative ϵ's. On the other hand, they describe how m can be found once the quotients in the Euclidean algorithm for

b_1 and b_2 are known. They finally prove that the number of equations plus the sum of the quotients is the same in both algorithms. *N. G. de Bruijn* (Delft).

Janković, Zlatko. Une démonstration de la formule de Bernoulli. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 7, 23–29 (1952). (Serbo-Croatian. French summary)

Moller, Raymond. Sums of powers of numbers having a given exponent modulo a prime. Amer. Math. Monthly 59, 226–230 (1952).

Let p be any prime and d any divisor of $p-1$; let n be any positive integer and let d' be the highest common factor of n and d. The author proves that the sum of the nth powers of all the distinct (mod p) numbers that belong to the exponent d is congruent, mod p, to $\varphi(d)\mu(d/d')/\varphi(d/d')$. His proof is based on the principle of cross-classification and the known result for $n=1$. Using his result he then proves that the sum of the nth powers of all the distinct numbers belonging, mod p, to any of the divisors of d is congruent, mod p, to d or to zero, according as n is or is not a multiple of d. H. S. Zuckerman adds the comment that the second of these theorems could be proved directly and the first one then deduced from it by means of the Moebius inversion formula. *H. W. Brinkmann* (Swarthmore, Pa.).

***Gel'fond, A. O. Rešenie uravenenii v celyh čislah. [The solution of equations in whole numbers.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 63 pp. 85 kopecks.

A booklet in the series, Popular Lectures in Mathematics.

Sándor, Gyula. Über die Anzahl der Lösungen einer Kongruenz. Acta Math. 87, 13–16 (1952).

Let N denote the number of solutions of the congruence $f(x)\equiv 0 \pmod{p^\alpha}$, where $f(x)$ is a polynomial of degree n and non-zero discriminant divisible by exactly p^δ. Then $N\leqq np^\delta$ and if $\alpha>\delta$, $N\leqq np^{\delta/2}$, N being independent of α in this case. *I. Niven* (Eugene, Ore.).

Swinnerton-Dyer, H. P. F. A solution of

$$A^5+B^5+C^5=D^5+E^5+F^5.$$

Proc. Cambridge Philos. Soc. 48, 516–518 (1952).

Two three-parameter families of solutions are given for the equation indicated in the title, each of which also satisfies $A+B+C=D+E+F$. *I. Niven* (Eugene, Ore.).

***Chudyniv-Bohun, Volodymyr. Solution of the Euler's problem.** Ukrainian Free Academy of Science. Series: Mathematics. Publ. no. 1. Privately printed, Regensburg, 1947. 20 pp.

The problem of the title is one that Euler proposed to Lagrange in 1770. It consists in finding 16 numbers A_{ij}, $i,j=1,2,3,4$, satisfying the 22 relations

$$\sum_{j=1}^{4}A_{ij}^2=\sum_{i=1}^{4}A_{ij}^2=\sum_{i=1}^{4}A_{ii}^2=\sum_{i=1}^{4}A_{i,5-i}^2=S,$$

$$\sum_{i=1}^{4}A_{ki}A_{ji}=\sum_{i=1}^{4}A_{ik}A_{ij}=0.$$

A two-parameter family of solutions of this problem is given. The parameters however are not free but must satisfy 11 other conditions. However, these latter conditions are fairly easy to meet. Many examples are worked out in which the elements A_{ij} are integers, complex numbers, quaternions, octonions, and sedecimions. *D. H. Lehmer.*

Kesava Menon, P. On Gauss's sums. J. Indian Math. Soc. (N.S.) 16, 31–36 (1952).

For integral values of a, x, M the function $F(a,x,M)$ is defined by the relation

$$F(a,x,M)=\sum_{m \pmod M}\exp\frac{2\pi i}{M}(am^2+xm),$$

where $\sum_{m \pmod M}$ indicates summation over a complete set of residues modulo M. Two properties of this function are established. The simpler is that if $N=aA^2+bB^2$ is prime to M, then

$$F(a,x,M)F(b,y,M)=F(N,Ax+By,M)F(Nab,aAy-bBx,M).$$

W. H. Gage (Vancouver, B. C.).

Pérez-Cacho, L. The function $E(x)$ (integral part of x) in the theory of numbers. Revista Mat. Hisp.-Amer. (4) 12, 36–40 (1952). (Spanish)

The author establishes by induction the well-known formulas

$$\sum_{x=1}^{n}\left[\frac{n}{x}\right]=\sum_{x=1}^{n}d(x),\quad \sum_{x=2}^{n}\phi(x)\left[\frac{n}{x}\right]=\tfrac{1}{2}n(n-1).$$

R. Bellman (Santa Monica, Calif.).

Perez-Cacho, L. The functions $y=\omega_k(n)$ in the theory of numbers. Study of the function $y=\omega_1(n)$. Memorias de Matemática del Instituto "Jorge Juan," no. 12, 34 pp. (1951). (Spanish)

Let $f(n)$ be any integral-valued arithmetic function such that $0<f(n)<n$ for $n\geqq 2$ and $f(1)=1$ or 0; let $n_1=n-f(n)$, $n_2=n_1-f(n_1)$, $\cdots$ and continue this process until the number $n_m=1$ is reached. The author then sets

$$\omega(n)=n_1+n_2+\cdots+n_m$$

and studies this function for various functions $f(n)$. Thus for $f(n)=n-\varphi(n)$, $\omega(n)$ is the function $P(n)$ already studied by the author [same Memorias, no. 7 (1948); these Rev. 11, 81]. In this paper the author studies three special cases: $\omega_1(n)=\omega(n)$ for $f(n)$ equal to the largest factor of n (distinct from n); $\omega_2(n)=\omega(n)$ for $f(n)=\varphi(n)$; $\omega_3(n)=\omega(n)$ for $f(n)=1$ when n is a prime or 1, $f(n)=2$ when n is composite. The function $\omega_1(n)$ is studied at greatest length; for example, it is proved that the only numbers for which $\omega_1(n)=n$ are those of the form $n=3\cdot 2^k$ and the only solutions of $\omega_1(n)=n-1$ are the numbers of the form $n=2^k$. A typical property of $\omega_2(n)$ is: $\omega_2(4n)=2\omega_2(2n)+1$. An explicit evaluation is given for $\omega_3(n)$. Tables are given for all three functions. *H. W. Brinkmann* (Swarthmore, Pa.).

Carlitz, L. Congruences for the coefficients of hyperelliptic and related functions. Duke Math. J. 19, 329–337 (1952).

The author points out the method of his previous paper [same J. 16, 297–302 (1949); these Rev. 10, 593] for deriving congruence properties of the coefficients of the Jacobi elliptic function $\mathrm{sn}(x,k^2)=\sum_1^\infty a_m x^m/m!$, where the rational number $l=k^2$ is integral (mod p), and p is an odd prime, fails in the hyperelliptic case. In the present paper this method is modified to apply not only to the hyperelliptic case but more generally to the class of functions $g(x)=\sum_1^\infty c_m x^m/m!$ such that the inverse function is of the form $\sum_1^\infty e_m x^m/m$,

where the c_m, e_m are integral (mod p) and $c_1=e_1=1$. It is first shown that

$$(1)\quad \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}c_p^{r-i}c_{m+i(p-1)}\equiv 0 \pmod{p^s},$$

where $s=[\frac{1}{2}(r+1)]$. This result is, however, weaker than the corresponding result in the elliptic case where the modulus p^s may be replaced by p^r for $m\geqq r\geqq 1$. A formula similar to (1) also holds for the coefficients of $g^h(x)$. Next the author puts $x/g(x)=\sum_0^\infty \beta_m x^m/m!$, and proves that $\beta^m(\beta^{p-1}-c_p)^r\equiv 0 \pmod{p^s}$, (where after expansion of the left member β^k is replaced by β_k) for $m\geqq r\geqq 1$ and $s=[\frac{1}{2}(r-1)]$. Finally a more general result is derived for the coefficients of $(x/g(x))^h$. *A. L. Whiteman* (Princeton, N. J.).

Erdös, P. On the greatest prime factor of $\prod_{k=1}^{x} f(k)$. J. London Math. Soc. 27, 379–384 (1952).

Let $f(x)$ be a polynomial with integral coefficients which is not the product of linear factors with integral coefficients. P_x denotes the greatest prime factor of $\prod_{k=1}^{x} f(k)$. The author shows that, for some $c>0$,

$$P_x > x\,(\log x)^{c\log\log\log x};$$

thus improving Nagell's inequality $P_x > cx\log x$ [Abh. Math. Sem. Univ. Hamburg 1, 179–194 (1922)]. The main lemmas are: (1) The number of positive integers $t\leqq x$ for which $f(t)$ is divisible by a number in the interval $(x/\log\log x, x)$ is greater than

$$cx\,(\log\log x)(\log\log\log x)/\log x;$$

(2) (due to Nagell) if $f(k)=A_kB_k$, where the prime factors of A_k are $\leqq x$, and those of B_k are $>x$, then we have

$$\sum_{k=1}^{x}\log A_k < x\log x + O(x).$$

The author states that he is able to prove, in a much more complicated way, that $P_x > x\exp((\log x)^c)$. *N. G. de Bruijn* (Delft).

Eljoseph, Nathan. Notes on a theorem of Lagrange. Riveon Lematematika 5, 74–79 (1952). (Hebrew. English summary)

The main theorem proved in this paper is as follows: A necessary and sufficient condition that it be possible to represent a Gaussian integer $a+bi$ as a sum of squares of Gaussian integers is that b be even. In the latter case $a+bi$ may be always represented as the sum of two such squares, except in the case where both a and b are twice odd numbers. In any case any number $a+2ci$ may be represented in an infinite number of ways as the sum of three Gaussian integers. When a Gaussian integer can be represented as the sum of two squares, the number of representations is finite and can be easily counted. The fundamental theorem of arithmetic for Gaussian integers is used for that purpose. (Author's summary.) *E. G. Straus* (Los Angeles, Calif.).

Cugiani, Marco. Sull'aritmetica additiva dei numeri liberi da potenze. Rivista Mat. Univ. Parma 2, 403–416 (1951).

Let $E(N)$ denote the number of representations of an integer N in the form $N=x^g+l_t$, where x is a square-free integer and l is a tth power-free integer. In this paper the following extension of a theorem of Roth [J. London Math. Soc. 22, 231–237 (1948); these Rev. 9, 499] is proved. If $g\geqq 2$ is fixed and $t=g$, then there exist positive constants γ_1, γ_2, γ_3 such that (1) $E(N)>\gamma_1 N^{1/g}(\log\log N)^{-1}$ for N sufficiently large; (2) $E(N)<\gamma_2 N^{1/g}(\log\log N)^{-1}$ for infinitely many values of N; (3) $E(N)>\gamma_3 N^{1/g}$ for infinitely many values of N. The methods employed are extensions in various directions of methods used by a number of writers among whom the following two may be cited: Ricci [Tôhoku Math. J. 41, 20–26 (1935)] and Rao [Proc. Indian Acad. Sci., Sect. A. 11, 429–436 (1940); these Rev. 2, 42]. *A. L. Whiteman* (Princeton, N. J.).

Cugiani, Marco. Sull'aritmetica dei polinomi di esponenziali a valori interi. Boll. Un. Mat. Ital. (3) 7, 38–43 (1952).

Let $F(y)=c_0y^n+c_1y^{n-1}+\cdots+c_n$ ($c_0\neq 0$, $c_n\neq 0$, $n\geqq 1$) be an irreducible polynomial with integral coefficients. Let P_x denote the largest prime divisor of the product $F(a)F(a^m)\cdots F(a^{xm})$ (a, x, m integers, $m\geqq 1$, $|a|\geqq 2$, $(a, c_n)=1$). The author proves that there exists a positive number γ independent of x such that $P_x>\gamma(x\log x)^{1/2}$ for sufficiently large x. The proof is based upon the following three lemmas in which the letter p stands for a prime. I. For $(k, p)=1$ the congruence $y^m\equiv k \pmod{p^s}$ possesses at most $2m$ solutions (mod p^s). II. The number $N(\xi)$ of solutions of the congruence $x^m\equiv kp^r \pmod{gp^s}$ (where $(k, p)=(g, p)=1$, $0<r<s$) which do not exceed a given real number ξ, satisfies the inequality $N(\xi)\leqq 2m(\xi p^{-s/m}+1)$. III. If x is a real number then $\sum_{p\leqq x} p=O(x^2/\log x)$. The author's theorem is an extension of a result previously obtained by G. Ricci [same Boll. 12, 222–228 (1933)]. *A. L. Whiteman* (Princeton, N. J.).

***Nečaev, V. I. Waring's problem for polynomials.** Trudy Mat. Inst. Steklov., v. 38, pp. 190–243. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

In this monograph Vinogradov's methods and results (including his estimate for Weyl sums) are applied to Waring's problem for polynomials. Let $f(x)$ be a polynomial of degree n with integral coefficients. The H.C.F. of such a polynomial is defined to be the greatest integer d which divides all values of $f(x)$ arising from integral x. Let $G(f)$ denote the least r with the property that every sufficiently large positive integer N is representable as

$$N=d^{-1}f(x_1)+\cdots+d^{-1}f(x_r)$$

with positive integral $x_1, \cdots, x_r$. The main problem is to determine or estimate $G(f)$; and there is also the same problem for $g(f)$, which is defined in the same way but omitting the words "sufficiently large". The present treatment is concerned primarily with values of $n\geqq 5$. The results are somewhat complicated to formulate, and it may suffice to mention one: if $r_0(f)$, defined below, satisfies $r_0(f)\geqq 10n^2\log n$, then $G(f)$ is either $r_0(f)$ or $r_0(f)+1$. Since there is an example, due to Hua, of a polynomial of degree $n\geqq 5$ for which $G(f)=2^n$ or 2^n-1 according as n is even or odd, it follows that the upper bound of $G(f)$ for f of degree n can be determined with a possible error of 1. The number $r_0(f)$ is defined by congruential considerations. Let $g(f, p^\beta)$ denote the least r for which the congruence

$$N\equiv d^{-1}f(x_1)+\cdots+d^{-1}f(x_r) \pmod{p^\beta}$$

is soluble for every integer N. Let d' be the H.C.F. of $f'(x)$; then r_0 is the greatest value of $g(f, p^\beta)$ for all prime factors p of d' and all positive integers β. Inequalities for $G(f)$ when $r_0(f)<10n^2\log n$ are also given but are less exact than that stated above. The author further proves that $G(f)\leqq 65$ when

$n=6$, and in view of Hua's example this upper bound cannot be in error by more than 1.

Chapters 1 and 2 are of an arithmetical character, and are concerned with $g(f, p^\beta)$. It is unfortunate that there is no explicit statement of the relation between the number $g_1(f)$ mentioned in the introduction and the number $g_0(f)$ discussed in chapter 2. Chapter 3 deals with the singular series. Chapter 4 quotes various results from Vinogradov's book [Trudy Mat. Inst. Steklov., v. 23 (1947); these Rev. 10, 599] and a recent paper [Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 199–214 (1950); these Rev. 12, 161]; also a few older results. Chapters 5 to 7 constitute the main body of the work. The author investigates the number of representations of Nd as

$$f(x_1)+\cdots+f(x_r)+u+u',$$

where $1\leqq x_i \leqq P$ and u and u' run through numbers of the form $f(\xi_1)+\cdots+f(\xi_k)$, where $\xi_1, \cdots, \xi_k$ are in intervals of successively lower orders of magnitude of the kind now usual in Waring's problem. The integral for the number of representations is split into basic and supplementary intervals in the usual way, and various choices of r and k are made according to the ranges of n and of $r_0(f)$. Chapter 8 is concerned with the polynomial $\varphi(x)=x(x+1)\cdots(x+n-1)$, and it is proved by a combination of elementary and analytical methods that $g(\varphi)<\frac{3}{2}n^2 \log n+6n \log n$.

H. Davenport (London).

Watson, G. L. A proof of the seven cube theorem. J. London Math. Soc. 26, 153–156 (1951).

Linnik's proof [Mat. Sbornik 12(54), 218–224 (1943); see also Pall, Canadian J. Math. 1, 344–364 (1949); these Rev. 5, 142; 11, 643] that every large positive integer is a sum of seven positive integral cubes, required some deep results on the representation of numbers by ternary quadratic forms. The present simpler proof uses, concerning ternary forms, only the fact that every $8u+3$ is a sum of three squares, but is based on an estimate for the number of primes in an arithmetic progression. This estimate uses (as also did Linnik's proof) Siegel's asymptotic formula for the class-number of binary quadratic forms. Using this estimate, it is shown that if X is sufficiently large, and $k<(\log X)^{100}$ and $(k, l)=1$, then there is a prime $p\equiv l \pmod{k}$ such that $X<p<1.01\,X$. This makes it possible, with $N=n-l^3$, to choose suitable primes p, q, r so that

$$8N=(4q^{18}+2r^{18})p^3+6q^6r^6p(8u+3),$$

and hence, putting $8u+3=x^2+y^2+z^2$,

$$8N=(q^6p+r^6x)^3+(q^6p-r^6x)^3+(q^6p+r^6y)^3+(q^6p-r^6y)^3 +(r^6p+q^6z)^3+(r^6p-q^6z)^3,$$

with positive even integers in the parentheses. *G. Pall.*

✓***Estermann, T. Introduction to modern prime number theory.** Cambridge Tracts in Mathematics and Mathematical Physics, no. 41. Cambridge, at the University Press, 1952. x+75 pp. $2.50.

The main purpose of this book is to give a complete account of Vinogradoff's famous theorem that every large odd positive integer can be represented as the sum of three primes [Doklady Akad. Nauk SSSR 15, 291–294 (1937); Mat. Sbornik N.S. 2(44), 179–195 (1937)]. It assumes some knowledge of the elementary theory of numbers, for which Hardy-Wright's treatise is quoted, and some of complex function theory. The exposition is very clear and accurate, and also in other respects reminds one of Landau's books. Problems, methods or results which do not directly fit in the author's presentation of Vinogradoff's theorem are completely neglected, and, therefore, the title of the book may be somewhat misleading. Nevertheless it gives a good impression of analytical number theory.

Chapter 1 presents a proof of de la Vallée Poussin's refinement of the prime number theorem

$$\pi(m)=\operatorname{li} m+O(m \exp(-c(\log m)^{\frac{1}{2}})).$$

Chapter 2 gives the much deeper analogous result for the primes in an arithmetical progression with emphasis on uniformity in k:

$$|\pi(m; k, l)-(\operatorname{li} m)/\varphi(k)|<Cm \exp(-(\log m)/200)$$

($(k, l)=1$, $k\leqq \log^u m$, C depending on u only), due to Walfisz. Chapter 3 gives Vinogradoff's theorem.

N. G. de Bruijn (Delft).

Chowla, Sarvadaman. The Riemann zeta and allied functions. Bull. Amer. Math. Soc. 58, 287–305 (1952).

This address, delivered to the American Mathematical Society, Dec. 28, 1949, gives a general review of the theory of the Riemann zeta-function, the prime-number problem, Dirichlet's L-functions, zeta-functions of an algebraic number-field, and L-functions of the type considered by A. Weil. A list of related unsolved problems is also given.

E. C. Titchmarsh (Oxford).

Faircloth, Olin B. On the number of solutions of some general types of equations in a finite field. Canadian J. Math. 4, 343–351 (1952).

The paper is concerned with the number of solutions of equations of the type

$$(*) \qquad c_1x_1^{m_1}+\cdots+c_sx_s^{m_s}=c \quad (c_i, c \in GF(p^n)).$$

The basic device employed is an expression for the number of non-zero solutions in terms of a generalized Jacobi-Cauchy function introduced by Vandiver [Proc. Nat. Acad. Sci. U. S. A. 36, 144–151 (1950); these Rev. 11, 330]. A number of general results are obtained that are too complicated to reproduce here. Among the special results may be mentioned a simple explicit formula for the number of non-zero solutions of (*) in the case $c_i=c=1$, $m_1=\cdots=m_s=m$, n even, $m\,|\,p^{in}+1$. *L. Carlitz* (Durham, N. C.).

Carlitz, L. The number of solutions of certain equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 38, 515–519 (1952).

The author begins by considering the equation (*) $x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}=\alpha$ where the unknowns x_i as well as the given number α are in a finite field $GF(p^n)$ and where a_i are positive integers with no common factor; this equation is solved by a simple change of variables and the number of its solutions is readily found. This fundamental idea is applied to various more general equations; thus it is possible to get a formula for the number of solutions of

$$x_1^{a_1}\cdots x_k^{a_k}=f(y_1, \cdots, y_r)$$

where y_j are additional unknowns in $GF(p^n)$ and $f(y_1, \cdots, y_r)$ is a polynomial with coefficients in $GF(p^n)$, provided a formula for the number of solutions of $f(y_1, \cdots, y_r)=\alpha$ is available for any value of α. Another generalization consists in replacing the left-hand side of (*) by a linear combination of such terms, with coefficients that do not vanish and belong to $GF(p^n)$. Well-known theorems enable the author to give explicit results in case the polynomial $f(y_1, \cdots, y_r)$ just mentioned is a quadratic form. Still another theorem deals

with an equation of the form $f(y_1, \cdots, y_r) = Q(z_1, \cdots, z_{2s})$, where Q is a quadratic form (with non-vanishing discriminant) in $2s^r$ variables. *H. W. Brinkmann.*

Skolem, Th. A remark on algebraic numbers. Norsk Mat. Tidsskr. 34, 14–17 (1952).

The following theorem is proved. Let R denote the field of rationals. Let $f(x)$ be the polynomial in R of lowest degree such that $f(\alpha)=0$ and let $g(x)$ be an arbitrary polynomial in R. Then a number $\beta \in R(\alpha)$ exists such that $g(\beta)=\alpha$ if and only if the polynomial $f(g(x))$ is divisible by a polynomial $h(x)$ which is irreducible in $R[x]$ and such that $\deg h(x) = \deg f(x)$. Several numerical examples of the theorem are given. *L. Carlitz* (Durham, N. C.).

Hua, Loo-Keng. On exponential sums over an algebraic number field. Canadian J. Math. 3, 44–51 (1951).

Let $f(x) = \alpha_k x^k + \cdots + \alpha_0$ be a polynomial of degree k with coefficients in an algebraic field K of degree n; let $\mathfrak{a}$ be the ideal $(\alpha_k, \cdots, \alpha_1)$ and $\mathfrak{b}$ the ground ideal (differente) of K. Write $\mathfrak{ab} = \mathfrak{r}/\mathfrak{q}$, where $\mathfrak{r}$ and $\mathfrak{q}$ are two relative-prime integral ideals, $S = S(f(x), \mathfrak{q}) = \sum_x e^{2\pi i \operatorname{tr}(f(x))}$, where x runs over a complete residue system $\bmod \mathfrak{q}$, and $\operatorname{tr}(f(x))$ denotes the trace of $f(x)$. It is proved that, for any positive ϵ, $S = O(N(\mathfrak{q})^{1-1/k+\epsilon})$, where the constant implied in O depends only on k, n, and ϵ. The proof is simpler than that of the special case for the field of rationals [Hua, J. Chinese Math. Soc. 2, 301–312 (1940); these Rev. 2, 347]. The proof is reduced to the case where $\mathfrak{q}$ is the power of a prime ideal $\mathfrak{p}$, and the evaluation of S for the case $\mathfrak{q} = \mathfrak{p}^l$ is shown to be obtainable from that with $l \leqq 2t+1$, where t is the highest exponent of $\mathfrak{p}$ dividing $\mathfrak{b}\mathfrak{a}^{-1}$, where

$$\mathfrak{b} = (k\alpha_k, (k-1)\alpha_{k-1}, \cdots, \alpha_1).$$

G. Pall (Chicago, Ill.).

Varnavides, P. The Euclidean real quadratic fields. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 111–122 (1952).

A real quadratic field $k = R(\sqrt{D})$, where R is the rational field and D is a positive square-free integer, is said to be Euclidean if Euclid's algorithm holds in it, i.e., if, for any non-zero integers α and β of k there is an integer ξ of k such that $|N(\alpha - \xi\beta)| < |N(\beta)|$, where $N(\ )$ denotes norm. It follows from a theorem of Davenport [Proc. London Math. Soc. (2) 53, 65–82 (1951); these Rev. 13, 15] that k is non-Euclidean if its discriminant d exceeds 2^{14}, where $d = D$ or $4D$ according as $D \equiv 1 \pmod 4$ or not. Various authors have treated cases of $d < 2^{14}$, with the combined "final" conclusion that k is Euclidean if and only if D has one of the 17 values $D = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97$. In this paper the author shows that k is Euclidean in the first 16 of these cases. He states that Rédei's claim [Math. Ann. 118, 588–608 (1942); these Rev. 6, 38] that $R(\sqrt{97})$ is Euclidean has been shown to be false by E. S. Barnes and H. P. F. Swinnerton-Dyer [paper in course of publication]. The method employed is an extension of one used by the author before [Quart. J. Math., Oxford Ser. 19, 54–58 (1948); these Rev. 9, 500], with further considerations required in the cases $D = 11, 19, 29, 41, 57, 73$. *R. Hull.*

Hochschild, G., and Nakayama, T. Cohomology in class field theory. Ann. of Math. (2) 55, 348–366 (1952).

Let K be an algebraic number field, $\mathfrak{K}$ a finite group of automorphisms of it, K^* its multiplicative group, J_K its idèle-group, and $C_K = J_K/P_K$ the idèle-class group, where P_K is the group of principal idèles, i.e. the image of K^* under the natural isomorphism. The importance of the cohomology theory of C_K was established by A. Weil [J. Math. Soc. Japan 3, 1–35 (1951); these Rev. 13, 439] from considerations much deeper than algebraic cohomology theory. In the present paper the cohomology groups of dimensions 1, 2, 3, in each of the three above mentioned groups, are studied by a technique of pure algebraic cohomology theory which though highly refined still demands much irksome computing.

The groups $H^1(\mathfrak{K}, K^*)$, $H^1(\mathfrak{K}, J_K)$, and $H^1(\mathfrak{K}, C_K)$ are all (1). Of course, $H^2(\mathfrak{K}, K^*)$ is isomorphic to the Brauer group of algebra classes with center k (= fixed field for $\mathfrak{K}$ in K); and $H^2(\mathfrak{K}, J_K)$ turns out to be isomorphic to the group of "ideal algebras" of MacLane and Schilling [Trans. Amer. Math. Soc. 50, 295–384 (1941); these Rev. 3, 102] so up to this point cohomology theory contributes nothing new to class field theory. But the group $H^2(\mathfrak{K}, C_K)$, which is cyclic of order n = degree K/k, is an important new development. It has a generator $u_{K/k}$, called the canonical cocycle, whose uniqueness comes from uniqueness of the norm residue symbol. Properties of $u_{K/k}$ are proved here by the method of Nakayama [Ann. of Math. 55, 73–84 (1952); these Rev. 13, 629], and the following global analogue of a result of Shafarevitch [C. R. (Doklady) Acad. Sci. URSS 53, 15–16 (1946); these Rev. 8, 250] is proved: Let $A \supset K \supset k$, A/K abelian with Galois group $\mathfrak{A}_K$, A/k normal with Galois group $\mathfrak{A}$. Then the Schreier factor set for $\mathfrak{K}$ in $\mathfrak{A}_K$, which describes $\mathfrak{A}$ as group extension of $\mathfrak{A}_K$, is the image of $u_{K/k}$ under the homomorphism of $H^2(\mathfrak{K}, C_K)$ into $H^2(\mathfrak{K}, \mathfrak{A}_K)$ induced by the reciprocity homomorphism of C_K onto $\mathfrak{A}_K$. For further important properties of $u_{K/k}$ see the papers by Weil and Nakayama already cited, also the review following.

Let $H_0^3(\mathfrak{K}, K^*)$ be the group of all 3-cocycles whose lift (to some extension field of K) is 1. It is proved that this group is cyclic of order n/n', where n' is the l. c. m. of the local degrees at all the prime spots, and its connection with "normal algebras" [Eilenberg and MacLane, Trans. Amer. Math. Soc. 64, 1–20 (1948); these Rev. 10, 5] is developed in the language of cocycles. The correspondingly defined groups $H_0^3(\mathfrak{K}, J_K)$ and $H_0^3(\mathfrak{K}, C_K)$ are (1). Finally, it is pointed out that all these facts, since they depend only on class field theory and abstract cohomology theory, hold also for fields of algebraic functions with a Galois field as field of constants. If, on the other hand, the field of constants is algebraically closed then the cohomology groups are all trivial.

Misprints (mostly supplied by Hochschild): p. 351, l. −17: for $x(1)$ read $\bar{x}(1)$; p. 352, l. 12: for f_1 read f_1^*; p. 355, diagram: read $\bar{j}\downarrow j\downarrow j_1\downarrow$ for $j\downarrow j\downarrow j_1\downarrow$; p. 356, l. −1: read "then" for "than"; p. 358, l. 6: read $c\epsilon k^*$ for $c\epsilon k^*$; p. 364, l. 8: this $G_{K,k}$ is not the whole Galois group but a certain subgroup of it. *G. Whaples* (Bloomington, Ind.).

Nakayama, Tadasi. Determination of a 3-cohomology class in an algebraic number field and belonging algebra-classes. Proc. Japan Acad. 27, 401–403 (1951).

There are two different ways of constructing a generator of $H_0^3(\mathfrak{K}, K^*)$ (for notation see preceding review). First, if $u_{K/k}$ is the canonical cocycle, its coboundary α defines a 3-cocycle in P_K, under the mapping $H^2(\mathfrak{K}, C_K) \to H_0^3(\mathfrak{K}, K^*)$, which generates the latter group. Second, if $\mathfrak{A}$ is any simple algebra with center K such that every $\sigma \epsilon K$ can be extended to an automorphism of $\mathfrak{A}$ then $\mathfrak{A}$ determines an element of $H_0^3(\mathfrak{K}, K^*)$ called its Teichmüller class; see Teichmüller

[Deutsch Math. 5, 138–149 (1940); these Rev. 2, 122] and Eilenberg and MacLane, loc. cit. in preceding review. Every element of $H_c^2(\mathfrak{K}, K^*)$ is also obtainable in this way. The following theorem explains the connection between these constructions: Let n_p be the p-degree of K/k, for each prime p of k, and let n' be the l. c. m. of all the n_p. Then $\mathfrak{A}$ has α as its Teichmüller class if and only if $(\mathfrak{A}/P)=(\mathfrak{A}/P')$ (Hasse invariants) for every pair of primes P, P' dividing the same p in k, and

$$\sum_p (\mathfrak{A}/P)n'/n_p \equiv -n'/(K:k) \pmod 1$$

where for each p, $(\mathfrak{A}/P)$ denotes the Hasse invariant at some $P|p$. (In the paper $(L:k)$ is a misprint for $(K:k)$.) The proof is only sketched here, details being promised in a later paper. It involves auxiliary cyclic fields and two rather complicated constructions. *G. Whaples.*

Nakayama, Tadasi. Note on an ordering theorem for subfields. Nagoya Math. J. 4, 125–129 (1952).

The following extensions of recent results of Tannaka [J. Math. Soc. Japan 3, 252–257 (1951); these Rev. 13, 726] are proved. 1) Let K be any field, k_1 and k_2 two subfields with K finite and separable (this condition can be somewhat weakened) over $k_1 \cap k_2$, and $M_i=\{\xi \mid \xi \epsilon K \text{ and } N_{K/k_i}\xi=1\}$. Then $k_1 \subsetneqq k_2$ if and only if $M_1 \supsetneqq M_2$. If $k_1 \subsetneqq k_2$ and k_1 has an infinite number of elements then the periods of elements of M_1M_2/M_1 are not bounded. 2) If k_i, K are p-adic number fields (finite over p-adic rationals) and A_i are the maximal abelian extensions of k_i, then $k_1 \subsetneqq k_2$ implies that KA_1A_2 is infinite over KA_2 and, in fact, there is a field X between KA_2 and KA_1A_2 such that the Galois group of KA_1A_2/X is topologically isomorphic to the additive group of p-adic integers. 3) If K, k_i are algebraic number fields and M_i the groups of idèles or idèle classes of norm 1, the analogue of 1) holds. 4) If K, k_i are algebraic number fields and $k_1 \subsetneqq k_2$, then for each rational prime p there is a field X, between KA_2 and KA_1A_2, such that the Galois group of KA_1A_2/X is topologically isomorphic to the additive group of p-adic integers. *G. Whaples* (Bloomington, Ind.).

Moriya, Mikao. Über die Restklassenkörper bewerteter perfekter Körper. Nagoya Math. J. 4, 15–27 (1952).

The author undertakes an analysis of the constructions of Hasse, F. K. Schmidt, S. MacLane, Teichmüller and Witt dealing with the structure of fields which are complete with respect to a discrete rank-one valuation. Suppose that K is a field complete with respect to a not necessarily discrete rank-one valuation. A mapping φ associating to each residue class $\mathfrak{m}$ in a subset $\mathfrak{M}$ of the residue class field $\mathfrak{K}$ uniquely one of its representatives $m=\varphi(\mathfrak{m})$ in K is termed a field (group) isomorphism if φ is so chosen that for a field (group) $\mathfrak{M}$ the images $\{m\}=M$ form a subfield (subgroup) of K. It is a known fact that such isomorphisms need not exist for arbitrary subsets $\mathfrak{M}$ if general valuations are considered. Suppose now, for the sake of brevity, that K and $\mathfrak{K}$ have the same prime characteristic. Then Zorn's Lemma implies immediately that K contains subfields K_0 which are maximal field isomorphic representations of some subfield $\mathfrak{K}_0$ of $\mathfrak{K}$, i.e. no proper extension of K_0 in K can serve as a field isomorphic representation for a subfield of $\mathfrak{K}$. (Completeness of K is not needed.) Application of Hensel's Lemma then shows that $\mathfrak{K}$ is (at most) a purely inseparable extension of the residue class field $\mathfrak{K}_0$ of a maximal K_0. (Only relative completeness in the sense of Ostrowski is needed.) If K is perfect then the following criterion for the existence of an isomorphic imbedding of $\mathfrak{K}$ in K is readily established: For each residue class $\mathfrak{c}\epsilon\mathfrak{K}$ let $\mathfrak{M}_i(\mathfrak{c})$ denote the set of all p^ith powers of all elements in $\mathfrak{c}^{p^{-i}}$ $(i\geqq 0)$, then a field isomorphism φ on $\mathfrak{K}$ into K exists if and only if $\bigcap_{i=0}^{\infty}\mathfrak{M}_i(\mathfrak{c})\neq\phi$ for all $\mathfrak{c}\epsilon\mathfrak{K}$. Furthermore, it is shown how the above mentioned results have to be rephrased so as to give group isomorphic imbeddings if K and $\mathfrak{K}$ have distinct characteristics. *O. F. G. Schilling* (Chicago, Ill.).

Kawada, Yukiyosi. On an arithmetic foundation for class-field theory. Sûgaku 1, 65–76 (1948). (Japanese)

In this paper the author gives a method for establishing arithmetically class-field theory. This method consists in translating the problem into relations between certain multiplicative groups of numbers of the ground field. Let k be a finite algebraic number field, and k^* the multiplicative group of the nonzero elements of k; let M be a finite set of primes containing all infinite primes and put $(\alpha)_M=\prod_{\mathfrak{x}\text{ non-}\epsilon M}\mathfrak{x}^{\nu_{\mathfrak{x}}(\alpha)}$ where $\nu_{\mathfrak{x}}(\alpha)$ is the $\mathfrak{x}$-order of α. The mapping $\alpha\rightarrow(\alpha)_M$ maps k^* homomorphically into the group of all ideals of k which are prime to all primes contained in M, and by choosing a suitable M, this mapping maps k^* onto this group. The kernel of this mapping is the group of all M-units; i.e. $\alpha\epsilon k^*$ such that $\nu_{\mathfrak{x}}(\alpha)=0$ for all $\mathfrak{x}$ non-ϵ M.

Let $\mathfrak{m}$ be an ideal-module, $\mathfrak{G}_{\mathfrak{m}}$ the group of all ideals which are prime to $\mathfrak{m}$, $S_{\mathfrak{m}}$ the group of all $\alpha\epsilon k^*$ such that $\alpha\equiv 1 \bmod \mathfrak{m}$ and $\mathfrak{S}_{\mathfrak{m}}$ the group of all principal ideals (α) generated by $\alpha\epsilon S_{\mathfrak{m}}$. If M contains all primes contained in $\mathfrak{m}$, then we have $k^*/E^MS_{\mathfrak{m}}\cong\mathfrak{G}_{\mathfrak{m}}/\mathfrak{S}_{\mathfrak{m}}$. Let K be a finite abelian extension of k, and M' the set of all prolongations of primes contained in M. By choosing a suitable M, the mapping $\mathrm{A}\rightarrow(\mathrm{A})_{M'}$ becomes also an onto-mapping. If $\mathfrak{m}$ contains all primes of k which ramify in K with sufficiently high exponents, then an essential point of the class-field theory lies in the equality $[\mathfrak{G}_{\mathfrak{m}}:\mathfrak{H}_{\mathfrak{m}}]=[K:k]$ where $\mathfrak{H}_{\mathfrak{m}}$ is the ideal group of k which corresponds to the extension K/k. The author shows that $[\mathfrak{G}_{\mathfrak{m}}:\mathfrak{H}_{\mathfrak{m}}]=[k^*:N_{K/k}K^*E^MS_{\mathfrak{m}}]$ and calculates the latter index. This calculation goes parallel to Chevalley's calculation [Ann. of Math. 41, 394–418 (1940); these Rev. 2, 38]. The proofs of the reciprocity theorem and the existence theorem are also done almost in the same way as in Chevalley's paper. In the last part of this paper, the author gives some remarks on the relation of this method with C. Chevalley's idèle theory, and gives a new proof of Grunwald's theorem on the norm residue symbol.
T. Tamagawa (Tokyo).

Tamagawa, Tsuneo. On the similarity of the theories of algebraic numbers and algebraic functions. Sûgaku 3, 65–75 (1951). (Japanese)

The author describes in detail with complete proofs the similarity of the theories of algebraic numbers and algebraic functions with respect to additive idèles, characters and differentials. These conceptions were developed by A. Weil [J. Reine Angew. Math. 179, 129–133 (1938)] and C. Chevalley [Ann. of Math. 41, 394–418 (1940); these Rev. 2, 38]. The fundamental result of this paper is the self-duality, i.e. the topological isomorphism of the group $\bar{k}$ of the additive idèles and the dual group $(\bar{k})^*$ of $\bar{k}$, both for the algebraic number field k and for the algebraic function field k over arbitrary coefficient field Ω.

(i) Let k be a number field of finite degree, and $M=\{P\}$ be the set of all the prime divisors P of k. Let k_P be the completion of k with respect to P. A vector $\bar{a}=(\alpha_P)$ $(P\epsilon M, \alpha_P\epsilon k_P)$ is called an additive idèle if the values

$v_P(a_P) \geqq 0$ for almost all P. We can introduce as usual a topology in the set $\bar{k}$ of all the additive idèles so that $\bar{k}$ is a locally compact topological ring. A character of $\bar{k}$ is a continuous homomorphism of k into the module $\mathfrak{R}_1$ (real numbers mod integers). The dual group $(\bar{k})^*$ of $\bar{k}$ is the topological group of all the characters of $\bar{k}$. Then $\bar{k}$ and $(\bar{k})^*$ are topologically isomorphic by the mapping $\bar{a} \rightarrow \bar{a} \cdot \omega$ $(\bar{a} \epsilon \bar{k})$, where ω is a special character and $\bar{a} \cdot \omega$ means the scalar multiplication. A differential of $\bar{k}$ is a character of $\bar{k}$ which annihilates k. Then the group L_k of all the differentials of $\bar{k}$ is 1-dimensional over k, i.e. any differential can be represented as $a \cdot \psi$ $(a \epsilon k)$ for a special differential ψ.

(ii) Let k be an algebraic function field over Ω. Then the corresponding topology of the group of additive idèles is the linear topology [see Lefschetz, Algebraic topology, Amer. Math. Soc. Colloq. Publ., vol. 27, New York, 1942, pp. 41–88; these Rev. 4, 84] and $\bar{k}$ is linearly locally compact. A character of $\bar{k}$ is a continuous homomorphism of $\bar{k}$ into Ω. The self-duality holds also in this case. This topology and the duality theorem can be applied to the proof of the theorem of Riemann-Roch [cf. T. Tamagawa, J. Fac. Sci. Univ. Tokyo, Sect. I. 6, 133–144 (1951); these Rev. 12, 855].

Y. Kawada (Princeton, N. J.).

Tamagawa, Tuneo. On unramified extensions of algebraic function fields. Proc. Japan Acad. 27, 548–551 (1951).

Suppose that L/K is a finite normal extension with the Galois group $\mathfrak{G} = \{\sigma, \cdots\}$ of the algebraic function field K with the algebraically closed coefficient field k. If $(\omega_1, \cdots, \omega_G) = \Omega'$ is a basis of the k-space $\mathfrak{L}_L$ of the differentials of the first kind of L, then each $\sigma \epsilon \mathfrak{G}$ gives rise to an endomorphism of $\mathfrak{L}_L$ which may be expressed by $\Omega^\sigma = A(\sigma)\Omega$. Chevalley and Weil determined in answer to a query raised by Hecke the representation-theoretic structure of $A(\sigma)$ if k has the characteristic 0. The author settles the corresponding problem for prime characteristic p for the special case of cyclic extensions of degree p^ν, apart from a combination with the obvious generalization of the Chevalley-Weil theorem for extensions whose degrees are relatively prime to p. It is shown that the representation $\sigma \rightarrow A(\sigma)$ is equivalent to the identity representation counted once and the regular representation counted $g-1$ times where g is the genus of K. In the proof the direct sum decomposition of the space Ω into σ-invariant subspaces, σ a generator of $\mathfrak{G}$, $\mathfrak{L}_1, \mathfrak{L}_2, \cdots, \mathfrak{L}_g$ of orders 1 for $\mathfrak{L}_1$ and p^ν for $\mathfrak{L}_2, \cdots, \mathfrak{L}_g$ is obtained as a consequence of module theory and the hypothesis of lack of ramification. Since the representation of σ by $\mathfrak{L}_i$ $(i \geqq 2)$ is of the type

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ & \ddots & & \\ & & \ddots & \\ & & & \\ & & 1 & 1 \end{bmatrix}$$

it follows that additive differentials like $\omega^* = \omega_{i,2}$ $(i \geqq 2)$ with $\omega^{*\sigma} = \omega^* + \omega_{i,1}$ and $\omega_{i,1} = \varphi(\omega^*)$ belonging to K are of special significance. The lack of ramifications implies that the set of images $\varphi(\omega^*)$ has at most dimension $g-1$. The assertion of the theorem then follows from the observation that the conjugates of the differentials ω_{i,p^ν} $(i \geqq 2)$ are linearly independent (because their traces relative to K equal $\omega_{i,1} \neq 0$) and thus form bases of the $\mathfrak{L}_i$ which gives rise to the regular representation.

O. F. G. Schilling (Chicago, Ill.).

Clarke, L. E. Non-homogeneous linear forms associated with algebraic fields. Quart. J. Math., Oxford Ser. (2) 2, 308–315 (1951).

Let k be an algebraic number field of degree n, with r real and $2s$ complex conjugates, $n = r + 2s$. Employing a basis $\omega_1^{(j)}, \cdots, \omega_n^{(j)}$ of the integers of the conjugate $k^{(j)}$ of k, $j = 1, \cdots, n$, the author considers n linear forms $L_j = \sum_\nu \omega_\nu^{(j)} x_\nu$ $(j = 1, \cdots, n)$, with the agreement that the forms are real for $j = 1, \cdots, r$, and conjugate complex for $j = v$ and $j = v + s$, $v = r + 1, \cdots, r + s$. The determinant of the n forms is $\sqrt{d}$, where d is the discriminant of k. Let $a_1, \cdots, a_n$ be any n complex numbers of which the first r are real and the others are conjugate complex in pairs, numbered as are the forms. The author's first theorem is that there exists a number μ_n, depending only on n, such that the inequality

$$(1) \qquad |(L_1 - a_1) \cdots (L_n - a_n)| \leqq \mu_n |d|^{n/2}$$

is solvable in rational integers $x_1, \cdots, x_n$. If $s = 0$, it has been known that the exponent $n/2$ may be replaced by $1/2$. The author gives an example showing that the exponent $n/2$ cannot be improved when $n = 2$, $r = 0$, $s = 1$. He says that it is probable that a smaller exponent will suffice in all other cases. His second theorem states that the exponent $n/2$ may be replaced by $n/2 - (n-2)/(n-1)$ when n is an odd prime. Theorem 3 is that for certain cubic fields the exponent $2/3$ suffices, while Theorem 4 is that there are infinitely many cubic fields for which this is best. Theorem 1 is proved by associating a positive definite quadratic form with the set of linear forms, and applying Minkowski's reduction theory to it. Refinements give the second theorem and special arguments give the third and fourth.

R. Hull.

Davenport, H. Linear forms associated with an algebraic number-field. Quart. J. Math., Oxford Ser. (2) 3, 32–41 (1952).

The inequality (1) of the preceding review is improved to the extent of replacing the exponent $n/2$ by $n/2(n-s)$. The author employs inequalities due to Minkowski and to Mahler for which he gives proofs. His method is a modification of a method devised by Siegel for the case $s = 0$. He discusses its scope in the last section. By a refinement like Clarke's, he obtains the further result that, if $s > 1$, and n is a prime, the exponent can be replaced by $n/2(n-s) - (s-1)/(n-1)(n-s)$. By the same method the author also proves: for any ideal $\mathfrak{A}$ in k, any α (integral or not) in k, there exists an integer η in k such that $\eta \equiv \alpha \pmod{\mathfrak{A}}$ and $|N\eta| < C_n(N\mathfrak{A})|d|^{n/2(n-s)}$, where N denotes norm.

R. Hull (Lafayette, Ind.).

Varnavides, P. On the product of three linear forms. Proc. London Math. Soc. (3) 2, 234–244 (1952).

The author proves the following theorem: Let L_1, L_2, L_3 be linear forms in u_1, u_2, u_3 with determinant 1 and let c_2, c_3 be any given real numbers. Then for any $\epsilon > 0$ we can satisfy

$$|L_1(L_2 - c_2)(L_3 - c_3)| < 1/9.1, \qquad |L_1| < \epsilon,$$

in integers u_1, u_2, u_3 not all zero, except when c_2, c_3 are both zero and L_1, L_2, L_3 are multiples of P, Q, R (see below) in some order. In these cases, the product $|L_1 L_2 L_3|$ has the minimum value 1/7 or 1/9. Here P, Q, R are defined to be $u_1 + \theta_i u_2 + \theta_i^2 u_3$, $i = 1, 2, 3$ where θ_i are the roots of the equation $t^3 + t^2 - 2t - 1 = 0$ or of $t^3 - 3t - 1 = 0$. The constant 9.1 is not the best possible constant but the excluded cases are essential. This result differs in character from that of Mordell for two forms $L_1, L_2 - c_2$ [J. London Math. Soc. 26,

93–95 (1951); these Rev. 13, 16] in that the constant is not the same as for the homogeneous approximation.

B. W. Jones (Boulder, Colo.).

Cassels, J. W. S. The inhomogeneous minimum of binary quadratic, ternary cubic and quaternary quartic forms. Proc. Cambridge Philos. Soc. 48, 72–86 (1952).

Cassels, J. W. S. Addendum to the paper, The inhomogeneous minimum of binary quadratic, ternary cubic, and quaternary quartic forms. Proc. Cambridge Philos. Soc. 48, 519–520 (1952).

Let (x) denote $(x^{(1)}, \cdots, x^{(n)})$ and let $f((x))$ be a real homogeneous form in (x) of degree n. The author's results are of the type: There exist real (x_0) such that $|f((x))| \geqq |\Delta|/\gamma$ for all $(x) \equiv (x_0)$ (mod 1), where Δ depends on f and γ is a positive constant depending only on n. For $n=2$, f indefinite, $f((x)) = f(x, y) = ax^2+2bxy+cy^2$, $\Delta = \sqrt{(b^2-ac)}$, $\gamma \sim< 48$ (see "Addendum" for error on p. 76 and other errors). Further results for $n=2$ are: If f represents 0, there are indenumerably many such incongruent (x_0); if a, b, and c are rational then (x_0) may be chosen rational; with $\gamma=87$, there exists a transcendental (x_0); for almost all (x_0): $\min |f((x))| = 0$, $(x) \equiv (x_0)$ (mod 1), provided f is not a multiple of a product of two linear forms with rational coefficients; the last proviso is necessary. For $n=3$ the factorable case only is dealt with: $f((x)) = L_1L_2L_3$, L's linear, L_1 real, L_2 and L_3 conjugate complex, $\Delta = \det(L_1, L_2, L_3) \neq 0$, $\gamma = 420$. Also for $n=4$ only the factorable case is dealt with: $f((x)) = L_1L_2L_3L_4$, L's linear, $L_2 = \bar{L}_1$, $L_4 = \bar{L}_3$, $\Delta = \det(L_1, \cdots, L_4)$, $\gamma = 5300$. In these cases, if the L's are conjugate linear forms of an algebraic field, then a rational (x_0) exists. The γ's for $n=3, 4$ are considerable improvements of results of Davenport [Acta Math. 84, 159–179 (1950); Trans. Amer. Math. Soc. 68, 508–532 (1950); these Rev. 12, 594], whose γ's are 8×10^{13} and 10^{123}, respectively. For $n=2$, the author's γ (see "Addendum") is less of an improvement of the $\gamma=128$ of Davenport [Proc. London. Math. Soc. (2) 53, 65–82 (1951); these Rev. 13, 15]. The author's methods differ from those of Davenport. They are related to those of Chabauty and Lutz [C. R. Acad. Sci. Paris 231, 887–888, 938–939 (1950); these Rev. 12, 483, 807].

R. Hull (Lafayette, Ind.).

Chalk, J. H. H. The minimum of a non-homogeneous binary cubic form. Proc. Cambridge Philos. Soc. 48, 392–401 (1952).

Let $f(x, y)$ be a binary cubic form with real coefficients and negative discriminant. For any real numbers x_0, y_0 there exist x, y satisfying $(x, y) \equiv (x_0, y_0)$ (mod 1) such that $8|f(x, y)| \leqq \max\{|f(1, 0)|, |f(0, 1)|, |f(1, 1)|, |f(1, -1)|\}$. It is proved geometrically by showing that, for each of certain lattices, the region $|x|(x^2+y^2) \leqq 1$ and its homothetic images at all lattice points cover the plane. Bambah has proved the similar theorem for forms of positive discriminant [same Proc. 47, 457–460 (1951); these Rev. 13, 114].

L. Tornheim (Ann Arbor, Mich.).

Sanov, I. N. A new proof of Minkowski's theorem. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 101–112 (1952). (Russian)

The theorem in question is that generally known as the Minkowski-Hlawka theorem; it was stated without proof by Minkowski and proved by Hlawka in a paper which appeared in 1943 [Math. Z. 49, 285–312 (1943); these Rev. 5, 201]. The author gives a proof of this theorem, based on another paper of his [Leningrad. Gos. Univ. Učenye Zapiski 111, 32–46 (1949)], which the reviewer has not been able to consult. The author states that this earlier paper was presented to a seminar at Leningrad in 1941, and that Lemma 1 in it represents an "insignificantly weaker" form of the theorem. There is no reference to the other proofs of the Minkowski-Hlawka theorem which have since been given [see, e.g., Rogers, Ann. of Math. 48, 994–1002 (1947); these Rev. 9, 270]. In view of these later proofs, the paper of Sanov is probably now only of historical interest.

H. Davenport (London).

Yûjôbô, Zuiman. On a theorem of Minkowski and its proof of Perron. Proc. Japan Acad. 27, 263–267 (1951).

The theorem of Minkowski meant is the following: For each couple of forms $L_1 = \alpha x+\beta y-\sigma$, $L_2 = \gamma x+\delta y-\tau$ $(\Delta = \alpha\delta-\beta\gamma \neq 0)$ there exists at least one lattice point (x, y) such that $|L_1L_2| \leqq \frac{1}{4}|\Delta|$. Using the method by which Perron proved the above theorem, the author proves that even an infinity of such lattice points with $|x| \to \infty$, $|y| \to \infty$ exist, assuming that $\delta \neq 0$, γ/δ is irrational, and $L_2 \neq 0$ for all lattice points (x, y). The case $L_1 = x$, $L_2 = \theta x - y - \vartheta$ has already been treated in the same way by the reviewer [Math. Ann. 116, 464–468 (1939); for further literature see this paper].

J. F. Koksma (Amsterdam).

✓ ***Davenport, H. Recent progress in the geometry of numbers.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 166–174. Amer. Math. Soc., Providence, R. I., 1952.

This address gives an account of a selection of some of the developments in the geometry of numbers since 1936. The results mentioned are explained clearly and appropriate references are given. A few rather deep problems are mentioned.

C. A. Rogers (London).

Val'fiš, A. Z. On lattice points in many-dimensional ellipsoids. X, XI, XII, XIII, XIV, XV. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 15, 275–296, 297–322 (1947); 16, 169–213, 215–230 (1948); 17, 245–258, 259–279 (1949). (Russian. Georgian summary)

[Communication IX (with the author's name spelled Walfisz) appeared in the Trav. Inst. Math. Tbilissi 10, 111–160 (1941); these Rev. 4, 132]. The number of lattice points in a k-dimensional sphere $z_1^2+\cdots+z_k^2 = x$ is given by

$$(1) \qquad A_k(x) = \sum_{j_1^2+\cdots+j_k^2 \leq x} 1 = \sum_{0 \leq n \leq x} r_k(x),$$

$j_1, \cdots, j_k$ being integers. This number is approximated by the area of the sphere, $V_k(x) = [\pi^{k/2}/\Gamma(\frac{1}{2}k+1)]x^{k/2}$, with an error $P_k(x)$,

$$(2) \qquad P_k(x) = A_k(x) - V_k(x).$$

In paper X the author establishes the following two results for $P_k(x)$.

$$(\mathrm{A}) \quad P_k(x) = 2M_k\{(1-2^{-\frac{1}{2}k})\zeta(\tfrac{1}{2}k)\}^{-1}\Psi_k(x)x^{\frac{1}{2}k-1} + Bx^{\frac{1}{4}(k-2)},$$

for $k \equiv 0$ (mod 4),

$$(\mathrm{B}) \quad P_k(x) = 2M_k\{L(\tfrac{1}{2}k)\}^{-1}\Psi_k(x)x^{\frac{1}{2}k-1} + Bx^{\frac{1}{4}(k-2)}$$

for $k\equiv 2 \pmod 4$, where

$$\zeta(x)=\sum_{n=1}^{\infty} n^{-x}, \qquad L(x)=\sum_{u=1}^{\infty}(-1)^{\frac{1}{2}(u-1)}u^{-x},$$

$$M_k=\pi^{\frac{1}{2}k}/2\Gamma(\tfrac{1}{2}k), \quad \psi(z)=z-[z]-\tfrac{1}{2},$$

$$\text{(3)}\quad \Psi_k(x)=-\sum_{u=1}^{\infty}u^{1-\frac{1}{2}k}\psi\left(\frac{x}{u}\right) +(-1)^{k/4}\sum_{\substack{n=1\\ n\equiv 0 \,(\mathrm{mod}\, 2)}}^{\infty} n^{1-\frac{1}{2}k}\left\{\psi\left(\frac{x}{n}\right)-2\psi\left(\frac{x}{2n}\right)\right\},$$

when $k\equiv 0 \pmod 4$, and

$$\text{(4)}\quad \Psi_k(x)=-\sum_{u=1}^{\infty}(-1)^{\frac{1}{2}(u-1)}u^{1-\frac{1}{2}k}\psi\left(\frac{x}{u}\right) +(-1)^{(k-2)/4}\sum_{n=1}^{\infty}(2n)^{1-\frac{1}{2}k}\left\{\psi\left(\frac{x}{n}\right)-\psi\left(\frac{x}{2n}\right)-2\psi\left(\frac{x-n}{4n}\right)\right\},$$

when $k\equiv 2 \pmod 4$. The results are obtained by replacing $r_k(n)$ in the definition of $A_k(x)$ by known expressions due to Hardy [see Proc. Nat. Acad. Sci. U. S. A. 4, 189–193 (1918); Trans. Amer. Math. Soc. 21, 255–284 (1920), sects. 3.1 and 5], to give

$$\text{(5)}\quad P_k(x)=2M_k\Bigg\{-\psi(x)x^{\frac{1}{2}k-1} +\sum_{l=2}^{\infty}\sum_{\substack{h=1\\ (h,l)=1}}^{l}\left(\frac{S_{h,l}}{l}\right)^k\sum_{n\leq x}n^{\frac{1}{2}k-1}e\left(-\frac{nh}{l}\right)\Bigg\}+Bx^{\frac{1}{2}(k-2)}.$$

Here the $S_{h,l}$ are Gauss sums and $e(z)=e^{2\pi iz}$. (A) and (B) then follow after lengthy manipulations of the sums in (5). Further, defining

$$\Psi_k=\limsup_{n\to\infty}\Psi_k(n), \qquad \psi_k=\liminf_{n\to\infty}\Psi_k(n),$$

$$P_k=\limsup_{n\to\infty}\frac{P_k(n)}{M_kn^{\frac{1}{2}k-1}}, \quad \rho_k=\liminf_{n\to\infty}\frac{P_k(n)}{M_kn^{\frac{1}{2}k-1}},$$

the author proves that for $k\equiv 0 \pmod 8$

$$\text{(6)}\quad \Psi_k=\tfrac{1}{2}\zeta(\tfrac{1}{2}k-1),$$

$$\text{(7)}\quad \psi_k=-\tfrac{1}{2}\zeta(\tfrac{1}{2}k-1)+(1-2^{-\frac{1}{2}k})\zeta(\tfrac{1}{2}k),$$

and hence that

$$\text{(8)}\quad P_k=\frac{\zeta(\frac{1}{2}k-1)}{(1-2^{-\frac{1}{2}k})\zeta(\frac{1}{2}k)}, \quad \rho_k=2-P_k.$$

Paper XI treats the case $k\equiv 4 \pmod 8$. Here it is only possible to prove that

$$\text{(9)}\quad 0\leqq\Psi_k-\tfrac{1}{2}\zeta(\tfrac{1}{2}k-1)+\theta_k\leqq 1.04(2^{-\frac{1}{2}k(k-1)}),$$

$$\text{(10)}\quad -1.04(2^{-\frac{1}{2}k(k-1)})\leqq\psi_k-(1-2^{-\frac{1}{2}k})\zeta(\tfrac{1}{2}k) +\tfrac{1}{2}\zeta(\tfrac{1}{2}k-1)-\theta_k\leqq 0,$$

where θ_k is given as an infinite series. (9) and (10) are used to obtain analogous results for P_k and ρ_k. The still more difficult case $k\equiv 2 \pmod 4$ is discussed in paper XII. Here the results are less precise. In paper XIII a shorter derivation is given for (A) and (B) based upon results of Z. Suetuna [J. Fac. Sci. Univ. Tokyo. Sect. I. 1, 249–283 (1926)]. A third and still shorter derivation of (A) and (B) is given in paper XIV based upon results of V. Jarnik [Math. Z. 30, 768–786 (1929)]. Paper XV again treats the cases $k\equiv 0$ (mod 4) and $k\equiv 2$ (mod 4) and gives more precise expressions for $P_k(x)$ which contain not only a term in $x^{\frac{1}{2}k-1}$ but also a term in $x^{\frac{1}{2}k-2}$ and with an error $O(x^{\frac{1}{2}(k-5)})$. For example, it is shown that for $k\geqq 6$ and $k\equiv 0 \pmod 2$,

$$P_{2k}(x)=\frac{\pi^k}{\Gamma(k)}\{(1-2^{-k})\zeta(k)\}^{-1} \times(\Psi_{2k}(x)x^{k-1}+\tfrac{1}{2}(k-1)\Pi_{2k}(x)x^{k-2})+Bx^{k-5/2},$$

where, in this case,

$$\Pi_{2k}(x)=\sum_{u=1}^{\infty}u^{2-k}\psi_1\left(\frac{x}{u}\right)+\sum_{g=2}^{\infty}g^{2-k}\left\{4\psi_1\left(\frac{x}{2g}+\frac{k}{4}\right)-\psi_1\left(\frac{x}{g}\right)\right\},$$

with $\psi_1(z)=(z-[z])^2-(z-[z])+\frac{1}{6}$. A similar expression is given for $P_{2k}(x)$ with $k\equiv 1 \pmod 2$. *W. H. Simons.*

Ankeny, N. C., and Rogers, C. A. A condition for a real lattice to define a zeta function. Proc. Nat. Acad. Sci. U. S. A. 37, 159–163 (1951).

Let x_i be real linear forms in u_j, i.e.

$$\text{(1)}\quad x^{(i)}=\sum_{j=1}^{n}u_jw_j^{(i)} \quad (i=1, 2, \cdots, n);$$

$w_j^{(i)}$ is real; the determinant $\Delta=|w_j^{(i)}|>0$; $P(X)=\prod_{i=1}^n x^{(i)}$. Let Λ denote the lattice consisting of all points in (1) where $u_1, \cdots, u_n$ take on all integer values. Suppose that there is no point X of Λ other than the origin 0 with $P(X)=0$, and suppose that $P(X)$ assumes only a finite number of different values α with $-\Delta\leqq\alpha\leqq\Delta$. Then it is shown, that the product $P(X)$ can be expressed in the form

$$P(X)=w\prod_{i=1}^{n}\left(\sum_{j=1}^{n}u_jw_j^{(i)}\right)$$

where $w_j^{(1)}$ $(j=1, \cdots, n)$ are algebraic integers in a field of degree n, and $w_j^{(i)}$ $(j=1, \cdots, n;\ i=2, \cdots, n)$ are their $(n-1)$ different algebraic conjugates. So one of the linear forms $\sum_{j=1}^n u_jw_j^{(1)}$ arises from a ring in an algebraic number field, and the other $(n-1)$ linear forms are the $(n-1)$ different conjugates of the first linear form. Hence, the product is essentially the norm of all numbers in an order of an algebraic number field, which clearly takes on only a finite number of values in any finite interval.

S. C. van Veen (Delft).

Bochner, S. Some properties of modular relations. Ann. of Math. (2) 53, 332–363 (1951).

A modular relation is a relation resembling an identity where

$$\text{(1)}\quad \Phi(x)=x^{-\delta}\Psi(1/x) \quad (\mathrm{Re}\,(x)>0)$$

where $\Phi(x)=\sum_0^\infty a_me^{-\lambda_mx}$, $\Psi(x)=\sum_0^\infty b_ne^{-\mu_nx}$, $\delta>0$, $\lambda_m\geqq 0$, $\mu_n\geqq 0$, sometimes $0<\lambda_1<\lambda_2\cdots\to\infty$, $0<\mu_1<\mu_2\cdots\to\infty$. In classical results $\lambda_m=m\lambda^*$, $\mu_n=n\mu^*$. In this case the relation (1) is equivalent to the functional equation

$$\text{(2)}\quad \Gamma(\delta-s)\psi(\delta-s)=\Gamma(\delta)\varphi(s)=\chi(s),$$

where $\varphi(s)=\sum_1^\infty a_m/\lambda_m^s$, $\psi(s)=\sum_1^\infty b_n/\mu_n^s$. In section I (Functional Equations) the main result is a generalisation of (2). Theorem 4: For general λ_m and μ_n, (2) implies

$$\text{(3)}\quad \sum_1^{\infty}a_me^{-\lambda_mx}-x^{-\delta}\sum_1^{\infty}b_n\exp(-\lambda_m/x)=P(x),$$

where $P(x)$ is a "residual" function $(2\pi i)^{-1}\int_C\chi(s)x^{-s}ds$, the

integral taken over a bounded curve or curves C, encircling all unramified singularities of $\chi(s)$. And conversely, any relation (3) with any residual function $P(x)$ leads back to an equation (2). By introducing Laplace-Stieltjes integrals, (3) is to be replaced by

$$(4) \qquad \int_0^\infty e^{-\lambda x} dA(\lambda) = x^{-\delta} \int_0^\infty \exp(-\mu/x) dB(\mu) = P(x)$$

where $A_\lambda = \sum_{\lambda_m<\lambda} |a_m|$, $B(\mu) = \sum_{\mu_n<\mu} b_n$. Usually the function $\chi(s)$ will have only a finite number of unramified singularities. In this case, by introducing the indefinite integrals

$$U(\lambda) = \frac{1}{2\pi i}\int_0^\lambda d\lambda \int_{C_1} \varphi(s)\lambda^{s-1} ds; \quad V(\mu) = \int_0^\mu d\mu \int_{C_2} \psi(s)\mu^{\delta-s-1} ds$$

(C_1 enclosing the singularities of $\chi(s)$ in the half plane $\sigma>0$, and C_2 those in the left-hand plane $\sigma<\delta$) and putting $R(\lambda)=A(\lambda)+U(\lambda)$; $S(\mu)=B(\mu)+V(\mu)$, the relation (4) assumes the "unified" appearance

$$\int_0^\infty e^{-\lambda x} dR(\lambda) = x^{-\delta}\int_0^\infty \exp(-\mu/x) dS(\mu)$$

in which the residual part $P(x)$ has been assimilated to the modular part proper.

In Section II (Summation Formulas) the author shows that under general conditions the "summation" formula

$$\int_0^\infty f(\lambda) dR(\lambda) = \int_0^\infty g(\mu) dS(\mu)$$

holds. Here $f(\lambda)$ is an arbitrary function in $0 \leqq \lambda < \infty$ and $g(\mu)$ is the Hankel transform

$$g(\mu) = \mu^{-\frac{1}{2}(\delta-1)} \int_0^\infty J_{\delta-1}\{2(\mu\lambda)^{\frac{1}{2}}\}\lambda^{\frac{1}{2}(\delta-1)} f(\lambda) d\lambda.$$

For instance, the Riesz kernel

$$f(\lambda) = \begin{cases} (1-\lambda)^\gamma & 0<\lambda<1 \\ 0 & 1 \leqq \lambda < \infty \end{cases}$$

falls under these conditions for $\gamma \geqq 2\beta-\delta-\frac{1}{2}$ and $\gamma>0$ simultaneously, where $\beta>0$ is a constant such that $\int_1^\infty \mu^{-\beta}|dS(\mu)|<\infty$. Then

$$\frac{1}{\Gamma(\gamma+1)}\int_0^x (x-\lambda)^\gamma dR(\lambda)$$

$$= \int_0^\infty (x/\mu)^{\frac{1}{2}(\delta+\gamma)} J_{\delta+\gamma}\{2(x\mu)^{\frac{1}{2}}\} dS(\mu), \quad 0<x<\infty.$$

In section III some supplementary remarks are made. In section IV (Modular Exponential Sums) many general results are obtained. We mention in particular, in the case of the modular relation (1) and under general assumptions: The series

$$\sum_1^\infty a_m \lambda_m^{l} e^{-2t\lambda_m^{\frac{1}{2}}}, \quad l \lesseqqgtr 0$$

and

$$\sum_0^\infty a_m (t^2/\lambda_m)^{\frac{1}{2}(\delta+\gamma)} J_{\delta+\gamma}(2t\lambda^{\frac{1}{2}}), \quad \gamma \lesseqqgtr 0$$

are Abel summable for $t^2>0$, if $t^2 \neq \mu_k$ $(k=1, 2, \cdots)$. In an addendum the author shows how the non-arithmetical part of Hecke's reasoning in regard to the functional equation for zeta functions in algebraic fields can be formalized and technically generalized. *S. C. van Veen* (Delft).

Obrechkoff, Nikola. Sur l'approximation diophantique des formes linéaires pour des valeurs positives des variables. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 343–356 (1950). (Bulgarian. French summary)

The present paper contains the detailed proofs of the results announced, and proved in outline, in an earlier note [Doklady Akad. Nauk SSSR (N.S.) **73**, 21–24 (1950); these Rev. **12**, 163], together with a further slight generalization. The doubt concerning one particular point, expressed in the review of the earlier note, is removed by the detailed proof now given. *H. Davenport* (London).

Hofreiter, Nikolaus. Über die Approximation von komplexen Zahlen durch Zahlen des Körpers $K(i)$. Monatsh. Math. **56**, 61–74 (1952).

The author proves: For each complex number ξ at least one couple of integers p, $q \neq 0$ in $K(i)$ exists such that $(p, q)=1$, $|\xi - p/q| \leqq \sqrt{(2-\sqrt{3})}/|q|^2$; the constant $\sqrt{(2-\sqrt{3})}$ is best possible as the example $\xi = \frac{1}{2}(1+i\sqrt{3})$ shows. The analogous result and more general theorems in the real case have been proved by Prasad [J. London Math. Soc. **23**, 169–171 (1948); these Rev. **10**, 513]. *J. F. Koksma* (Amsterdam).

Poitou, Georges, et Descombes, Roger. Sur certains problèmes d'approximation. II. C. R. Acad. Sci. Paris **234**, 1522–1524 (1952).

En utilisant la notation de la première partie [mêmes C. R. **234**, 581–583 (1952); ces Rev. **13**, 825] les auteurs en remarquant que le problème général équivaut à son cas particulier en faisant $a=1$, $b=0$, continuent à rechercher $k(\xi, s)=k(\xi, s, 1, 0)$ et $k(s)$ = la borne supérieure des nombres $k(\xi, s)$ pour tous ξ (s supposé fixe). Pour $3 \leqq s \leqq 10$ les auteurs donnent une liste de $k(s)$. Ces nombres sont isolés dans l'ensemble des valeurs de $k(\xi, s)$ et ne sont atteint que pour certains nombres quadratiques (dits critiques) ξ. Quant au cas $s=2$, on a $k(2)=1$, mais le nombre $k(2)$ n'est pas isolé dans l'ensemble des $k(\xi, 2)$. Les auteurs recherchent la distribution des nombres $C(s)=s^2/k(s)$ sur l'axe réel $(s=2, 3, 4, \cdots)$ et montrent $C(s) \geqq 2.5$, borne qui n'est pas la valeur la plus grande possible. Il y a une infinité de valeurs $C(s)<3$. Enfin les auteurs recherchent l'expression $\bar{K}(s)$, analogue à $K(s)$ qu'on trouve si dans la définition de $K(\xi, s)$ on admet aussi des valeurs négatives pour u et v (voir la note citée). *J. F. Koksma* (Amsterdam).

Steinberg, R., and Redheffer, R. M. Analytic proof of the Lindemann theorem. Pacific J. Math. **2**, 231–242 (1952).

The authors give a proof of the Lindemann-Weierstrass theorem: Let $\alpha, \beta, \cdots, \sigma$ be different algebraic numbers, let $a, b, \cdots, s$ be arbitrary algebraic numbers; then

$$ae^\alpha + be^\beta + \cdots + se^\sigma = 0$$

is possible only for $a=b=\cdots=s=0$. The paper is mainly expository. The method used is that in Hilbert's proof for the transcendence of e and π [Gesammelte Abhandlungen, Bd. 1, Springer, Berlin, 1932, pp. 1–4]. *J. Popken.*

ANALYSIS

Orloff, Konstantin. Sur un théorème des accroissements finis. Bull. Soc. Math. Phys. Serbie 3, no. 1–2, 71–73 (1951). (Serbo-Croatian. French summary)

Relative to the law of the mean,

$$(*)\qquad F(x)-F(y)=(x-y)F'(\xi),$$

R. Rothe [Math. Z. 9, 300–325 (1921)] obtained conditions on a given function $\xi(x, y)$ which are necessary for the existence of a function $F(t)$ satisfying (*) identically. R. Bojanić [Acad. Serbe Sci. Publ. Inst. Math. 3, 219–226 (1950); these Rev. 12, 483] extended the result of Rothe to give three necessary conditions on $\xi(x, y)$, which together are sufficient, for the existence of $F(t)$. These conditions are that $\xi(x, y)$ satisfy $\xi(x, y)=\xi(y, x)$ and $\xi(x, x)=x$, and that the expression $s/(pq)+(p-q)/[(y-x)pq]$, $y\neq x$, be a function of ξ alone. The author now reduces the number of conditions which are both necessary and sufficient to two; these are that $\xi(x, y)$ satisfy $\lim_{y\to x}\xi(x, y)=x$, and that, for $y\neq x$, the function $f(x, y)$, defined by $f(x, y)=(\xi_x+\xi_y)/[(x-y)\xi_y]$, satisfy

$$f(x, y)=\frac{f(x, \alpha)\exp\left[\int_y^x f(t, \alpha)dt\right]-f(y, \alpha)}{\exp\left[\int_y^x f(t, \alpha)dt\right]-1-f(y, \alpha)(x-y)},$$

where α is a constant for which $f(x, \alpha)\neq$const.

E. F. Beckenbach (Los Angeles, Calif.).

Darbo, Gabriele. Una estensione del secondo teorema della media. Ann. Scuola Norm. Super. Pisa (3) 5, 151–160 (1951).

The author generalizes the second mean value theorem for integrals of functions of one variable, as follows: Let $Q(x, y)$, defined in $a\leqq x\leqq b$, $c\leqq y\leqq d$, be monotone non-decreasing with respect to x for each y, and summable with respect to y for each x; let $y(x)$, defined in $a\leqq x\leqq b$, be absolutely continuous and satisfy $c\leqq y(x)\leqq d$; and let $Q[x, y(x)]y'(x)$ be quasi-continuous and summable. Then there exists at least one value ξ in (a, b) for which $\int_a^b Q[x, y(x)]y'(x)dx=\int_{y(a)}^{y(\xi)}Q(a, \eta)d\eta+\int_{y(\xi)}^{y(b)}Q(b, \eta)d\eta$.

E. F. Beckenbach (Los Angeles, Calif.).

Sharma, Ambikeshwar. On the properties of $\theta(x, h)$ in Mazzoni's form of the mean-value theorem. Math. Student 19, 37–43 (1951).

The author considers the theorem of Mazzoni [Rend. Circ. Mat. Palermo 52, 44–57 (1928)]:

$$f(x+h)=f(x)+hf'(x)+\cdots$$
$$+\frac{h^n}{n!}f^{(n)}\left(x+\frac{h}{n+1}\right)+\frac{nh^{n+2}}{2(n+1)\cdot(n+2)!}f^{(n+2)}(x+\theta h),$$

and, for certain classes of functions, he obtains the limit of $\theta=\theta(x, h)$ as $h\to 0$. These are the same classes of functions for which Sokolowski [Tôhoku Math. J. 31, 177–191 (1929)] obtained analogous results relative to the θ in Taylor's theorem. The real function $f(x)$ is said to be of class $C_{n+2,\rho}$ $(\rho>-1, \rho\neq 0)$ provided $f^{(n+2)}(x)$ exists at each point in the interval $(0, a)$, $a>0$, and, for a suitable l, $\lim_{h\to 0}[f^{(n+2)}(h)-l]h^{-\rho}$ exists, is finite, and $\neq 0$. Special definitions are given for the limiting values $\rho=-1, 0, \infty$. It is shown that if $f(x)$ is of class $C_{n+2,\rho}$ then

$$\theta=\left(\frac{2}{n}\right)^{1/\rho}\left[\frac{n+1}{\binom{n+2+\rho}{n+2}}-\frac{n+2}{(n+1)\rho(\rho+1)(\rho+2)}\right]$$

for $\rho>-1$, $\rho\neq 0$, with limiting values for $\rho=-1, 0, \infty$. By means of convexity properties it is shown that, as a function of ρ, θ varies continuously for $-1\leqq\rho\leqq\infty$.

E. F. Beckenbach (Los Angeles, Calif.).

Stojakovitch, Mirko. Sur une généralisation de la formule de Cauchy. Bull. Soc. Math. Phys. Serbie 3, no. 1–2, 35–37 (1951). (Serbo-Croatian. French summary)

Generalizing the Cauchy formula

$$\frac{f(x_0+h)-f(x_0)}{\varphi(x_0+h)-\varphi(x_0)}=\frac{f'(x_0+\theta h)}{\varphi'(x_0+\theta h)},\quad 0<\theta<1,$$

the author points out that

$$(*)\qquad \frac{f(x_0+h)-T_{n-1}(f; x_0)}{\varphi(x_0+h)-T_{n-1}(\varphi; x_0)}=\frac{f^{(n)}(x_0+\theta h)}{\varphi^{(n)}(x_0+\theta h)},\quad 0<\theta<1,$$

where the T_{n-1} are the indicated Taylor polynomials of degree $n-1$. It is noted that (*) yields, relative to the finite Taylor series for $f(x_0+h)$, a form of the remainder involving φ.

E. F. Beckenbach (Los Angeles, Calif.).

Faragó, T. Über das arithmetisch-geometrische Mittel. Publ. Math. Debrecen 2, 150–156 (1951).

The arithmetic-geometric mean of two positive numbers a_0, b_0 is the common limit $M(a_0, b_0)$ of the two sequences $\{a_n\}$, $\{b_n\}$ generated by $a_{n+1}=\frac{1}{2}(a_n+b_n)$, $b_{n+1}=(a_nb_n)^{\frac{1}{2}}>0$. For these sequences, we have

$$(*)\qquad b_n\leqq b_{n+1}\leqq M(a_0, b_0)\leqq a_{n+1}\leqq a_n.$$

For complex numbers a_0, b_0, the author now considers the same sequences, except that now the condition $b_{n+1}>0$ is generalized to the restriction that b_{n+1} lie in the angle $\alpha_{n+1}=\measuredangle(a_n, 0, b_n)$, with $0\leqq\alpha_n\leqq\pi$. Clearly we do not now necessarily have $|b_n|\leqq|a_n|$; nevertheless, the following analogue of (*) is given:

$$|b_n|\cos\tfrac{1}{2}\alpha_n\leqq|b_{n+1}|\cos\tfrac{1}{2}\alpha_{n+1}$$
$$\leqq|M(a_0, b_0)|\leqq\tfrac{1}{2}[|a_n|+|b_n|]\leqq\tfrac{1}{2}[|a_{n+1}|+|b_{n+1}|].$$

Also, necessary and sufficient conditions on (a_0, b_0) and (a_0', b_0'), in order that $|M(a_0, b_0)|\leqq|M(a_0', b_0')|$, are given.

E. F. Beckenbach (Los Angeles, Calif.).

Fuchs, L. On mean systems. Acta Math. Acad. Sci. Hungar. 1, 303–320 (1950). (English. Russian summary)

The discussion by Kolmogoroff [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 12, 388–391 (1930)] and Nagumo [Jap. J. Math. 7, 71–79 (1930)] of necessary and sufficient conditions in order that a sequence of mean value functions M_n be quasi-arithmetic,

$$M_n(x_1, \cdots, x_n)=f^{-1}\left[\frac{f(x_1)+\cdots+f(x_n)}{n}\right],$$

has subsequently been extended by many authors in various directions. The author now gives a further algebraic generalization, defining operations having properties of mean values

in completely ordered systems; in particular, the nonalgebraic postulate of continuity is here replaced by an archimedian axiom. *E. F. Beckenbach* (Los Angeles, Calif.).

Gurin, L. On the permutability of averaging and transforming. Uspehi Matem. Nauk (N.S.) 7, no. 1(47), 155–158 (1952). (Russian)

The author considers a given function $y=f(x_1, \cdots, x_n)$, a set of "invariant curves"

$$w_i(x_1, \cdots, x_n)=c_i \ (i=1, \cdots, n-1),$$

and a transformation

$$(*)\quad \bar{x}_1=U(x_1, \cdots, x_n), \quad w_i(\bar{x}_1, \cdots, \bar{x}_n)=w_i(x_1, \cdots, x_n),$$

with suitable assumptions concerning differentiability and nonvanishing of Jacobians. He also defines an averaging process for sets $(x_{1j}, \cdots, x_{nj})$ $(j=1, \cdots, N)$ and values t_j satisfying $\sum_{j=1}^N t_j=1$, by

$$(**)\quad f(x_{1m}, \cdots, x_{nm})=\sum_{j=1}^N t_j f(x_{1j}, \cdots, x_{nj}),$$
$$w_i(x_{1m}, \cdots, x_{nm})=c_{i0},$$

where the c_{i0} are fixed constants and the subscript m is used to denote the mean. It is shown that, for arbitrary $(x_{1j}, \cdots, x_{nj})$, there are values $(\bar{x}_{ij}, \cdots, \bar{x}_{nj})$ and $(\bar{x}_{1m}, \cdots, \bar{x}_{nm})$ satisfying (*) and (**) simultaneously if and only if there exists a function $z_1(x_1, \cdots, x_n)$ such that $f=Az_1+B$, $\bar{z}_1=Cz_1+D$, $\bar{w}_i=w_i$, where A, B, C, D are constants.

E. F. Beckenbach (Los Angeles, Calif.).

Nikolaev, V. F. On some interpolation processes. Doklady Akad. Nauk SSSR. (N.S.) 84, 441–444 (1952). (Russian)

In the author's terminology, an interpolation process consists of a triangular matrix of nodes $x_k^{(n)}$ and a set of cosine polynomials $I_n^{(m)}(x, f)$ coinciding with $f(x)$ at the points $x_k^{(n)}$, where $f(x)$ is continuous in $[0, \pi]$, $I_n^{(m)}$ is of order $n+m$, and m may depend on n. The author shows by examples of a rather general kind that the norm of the operator $I_n^{(m)}$ may become infinite even when the ratio $(n+m)/(n+1)$ of the order of the interpolating polynomial to the number of nodes is made to become infinite arbitrarily rapidly.

R. P. Boas, Jr. (Evanston, Ill.).

Schoenberg, I. J. On Pólya frequency functions. I. The totally positive functions and their Laplace transforms. J. Analyse Math. 1, 331–374 (1951). (English. Hebrew summary)

This paper contains a detailed exposition of some of the results previously announced by the author [Proc. Nat. Acad. Sci. U. S. A. 33, 11–17 (1947); 34, 164–169 (1948); these Rev. 8, 319; 9, 415]. Part II has already appeared [Acta Sci. Math. Szeged 12, Pars B, 97–106 (1950); these Rev. 12, 23]. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

San Juan, Ricardo. Errata: Some noteworthy asymptotic developments. Revista Mat. Hisp.-Amer. (4) 12, 63–65 (1952). (Spanish)

See same Revista (4) 11, 65–110 (1950); these Rev. 13, 214.

Škol'nik, A. G. Linear inequalities. Doklady Akad. Nauk SSSR (N.S.) 70, 189–192 (1950). (Russian)

The author gives some theorems on the consistency and independence of a system of linear inequalities, and on the boundedness of the solution set, in terms of the signs of the minors of the matrix of coefficients and free terms. The theorems are equivalent to the criteria of Motzkin [Dissertation, Basel, 1933, Jerusalem, 1936, pp. 49–50] for homogeneous systems. *J. M. Danskin.*

Theory of Sets, Theory of Functions of Real Variables

Wakulicz, Antoni. Sur les sommes de quatre nombres ordinaux. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 42 (1949), 23–28 (1952). (French. Polish summary)

It is shown that $k\leqq 13$ is a necessary and sufficient condition for the existence of four ordinal numbers whose sum assumes precisely k distinct values when the terms are permuted in all possible ways. Related results are discussed in a subsequent paper of the same author [Fund. Math. 36, 254–266 (1949); these Rev. 12, 14]. *F. Bagemihl.*

Neumer, Walter. Zum Beweis eines Satzes über die Polynomdarstellung der Ordnungszahlen. Math. Z. 55, 399–400 (1952).

Let the normal form of an ordinal number α be $\omega^{\alpha_0}a_0+\omega^{\alpha_1}a_1+\cdots+\omega^{\alpha_n}a_n$. Call $\alpha_0, \alpha_1, \cdots, \alpha_n$ the exponents of first rank of α, call their exponents of first rank the exponents of second rank of α, etc. Kaluza, Jr. [Math. Ann. 122, 321–322 (1950); these Rev. 12, 626] demonstrated with the aid of graph theory the existence of a natural number $k(\alpha)$ such that every exponent of α of rank $k(\alpha)$ is either an ϵ-number or zero. This is proved directly in the present note.

F. Bagemihl (Rochester, N. Y.).

Sierpiński, Wacław. Sur les bases dénombrables de la famille de tous des ensembles linéaires dénombrables. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 42 (1949), 182–184 (1952). (French. Polish summary)

A l'aide de l'hypothèse du continu Mazur [mêmes C. R. 31, 102–103 (1938); aussi Sierpiński, Fund. Math. 31, 259–261 (1938)] a prouvé l'existence d'une suite S dénombrable d'ensembles linéaires tels que chaque ensemble linéaire dénombrable soit limite d'une suite partielle de S; dans la présente note l'A. montre (sans se servir de l'hypothèse du continu) que les termes de S ne peuvent être ni mesurables L ni pourvus de la propriété de Baire (au sens large).

Đ. Kurepa (Zagreb).

Fan, Ky. Note on a theorem of Banach. Math. Z. 55, 308–309 (1952).

Banach's formulation [Fund. Math. 6, 236–239 (1924)] of the Schröder-Bernstein theorem on antisymmetry of cardinal order is generalized as follows. Let $A_1, A_2, \cdots, A_n$ $(n\geqq 2)$ be finitely many infinite sets arranged in cyclic order so that $A_{n+1}=A_1$, and let t_i be a one-to-one transformation from A_i into A_{i+1}. Then each A_i can be decomposed into the union of two disjoint subsets B_i, C_i such that t_i transforms B_i onto C_{i+1}. [Remark. The conclusion could just as well have been proved for: $n\geqq 1$, A_i arbitrary, t_i single-valued.] *W. Gustin* (Bloomington, Ind.).

Bourbaki, Nicolas. Sur le théorème de Zorn. Arch. Math. 2 (1949–1950), 434–437 (1951).

An adaptation of Zermelo's second proof of the well-ordering theorem leading to a demonstration of Zorn's lemma from the axiom of choice without overt intervention of well-ordering. *W. Gustin* (Bloomington, Ind.).

Fraïssé, Roland. Sur certains systèmes de relations qui généralisent les systèmes de base finie. C. R. Acad. Sci. Paris **234**, 1116–1119 (1952).

Further development of concepts introduced and considered in earlier papers [see especially same C. R. **230**, 1557–1559 (1950); these Rev. **12**, 14]. Automorphisms of multirelations are defined, as well as monotypic multirelations. *R. Arens* (Los Angeles, Calif.).

Dowker, C. H. A problem in set theory. J. London Math. Soc. **27**, 371–374 (1952).

Let $\mathfrak{A}$ be a proper ideal of subsets of a set E. Call a transformation T: $E\rightarrow\mathfrak{A}$ of type XY provided $x\epsilon T(y)$ or $y\epsilon T(x)$ for every two distinct points $x\epsilon X$, $y\epsilon Y$. Condition I: there exists a T of type EE. Condition II: for any two complementary sets X, Y there exists a T of type XY. Clearly I implies II. Problem: does II imply I? Define p as the minimal power of sets not in $\mathfrak{A}$ and q as the minimal power of complements of sets in $\mathfrak{A}$; since $\mathfrak{A}$ is a proper ideal, $p\leqq q$. It is shown that I holds if $p=q$ or if E is a countable union of sets from $\mathfrak{A}$, and that II implies I if $p\leqq\aleph_0$. Thus if II holds but not I, then $q>p>\aleph_0$, whereupon E has power $\geqq\aleph_2$. *W. Gustin* (Bloomington, Ind.).

Anzai, Hirotada. On an example of a measure preserving transformation which is not conjugate to its inverse. Proc. Japan Acad. **27**, 517–522 (1951).

By constructing an example of the type described in the title the author solves (negatively) a problem proposed by the reviewer and von Neumann [Ann. of Math. (2) **43**, 332–350 (1942); these Rev. **4**, 14]. The method of construction is similar to the one the author used once before [Osaka Math. J. **3**, 83–99 (1951); these Rev. **12**, 719]; the example is, in fact, a skew product transformation. *P. R. Halmos* (Chicago, Ill.).

Sunouchi, Gen-ichirô. On the sequence of additive set functions. J. Math. Soc. Japan **3**, 290–295 (1951).

It is shown that the Vitali-Hahn-Saks theorem can be generalized in the form: If $\mathfrak{M}$ is the family of measurable subsets of an abstract space M such that M is the sum of a denumerable number of measurable subsets M_n of finite measure, if $F_n(E)$ are completely additive (c. a.) and absolutely continuous (a. c.) and $\lim_n F_n(E)=F(E)$ for all E of $\mathfrak{M}$, then $F(E)$ is c. a. and a. c. and $F_n(E)$ are uniformly absolutely continuous in the sense that for every $\epsilon>0$, there exists a δ_ϵ, an m_ϵ and an n_ϵ such that if $\mu(E\cap M_i)<\delta_\epsilon$ for $i=1,\cdots,m_\epsilon$, then for all $n=n_\epsilon$ we have $|F_n(E)|<\epsilon$. The equivalence of weak and strong convergence in the case of the space l^1 of absolutely convergent series is deduced from this theorem. There is also a generalization of the Helly-Bochner theorem for a separable metric space M, viz., if $0\leqq\mu_n(E)\leqq K$ for all n and all measurable subsets of M, the Borel sets being assumed to be measurable, then there exists a subsequence μ_{n_k} and a measure μ, such that $\lim_k \mu_{n_k}(E)=\mu(E)$ for all sets E of continuity relative to μ, i.e. for which $\mu(E_f-E_0)=0$, where E_f is the closure of E and E_0 the interior of E. The result is applied to the convergence of integrals. *T. H. Hildebrandt.*

Rohlin, V. A. On the fundamental ideas of measure theory. Amer. Math. Soc. Translation no. **71**, 55 pp. (1952).

Translated from Mat. Sbornik N.S. **25(67)**, 107–150 (1949); these Rev. **11**, 18.

Ridder, J. The definite integral. Simon Stevin **29**, 1–12 (1952). (Dutch)

An elementary exposition of the definite integral contrasting the definitions by (a) sequences of successive subdivisions with maximum length of subinterval approaching zero, (b) Riemannian procedure, and (c) successive subdivision (Moore-Smith) procedure. *T. H. Hildebrandt.*

McShane, E. J., and Botts, T. A. A modified Riemann-Stieltjes integral. Duke Math. J. **19**, 293–302 (1952).

The modification involves assuming that in the Stieltjes integral $\int fdg$, the function g being monotone, g is defined on a nondegenerate bounded closed interval B in n-dimensional Euclidean space, and f is defined on a bounded set D containing B. In the approximating sums for a finite subdivision into rectangles I covering D, only the portions of the I intersecting B is used in the Δg, but the values of f may be in D. The limit is the usual one for Riemann integration, viz. as the maximum diameter of the subdivisions I approaches zero. The integral becomes the usual Riemann-Stieltjes integral if D is B. The integral thus depends on D, f, B and g, denoted by $S(D, f; B, g)$, but in a sense is independent of D as D may be replaced by a smaller set D' containing B without affecting the existence or value of the integral. In addition to a suitably modified integration-by-parts theorem for the case when B is a linear interval, it is emphasized that four properties of Riemann integration can be proved for this integral, viz.: (1) the existence of the integral is equivalent to the equality of an upper and lower integral, the latter utilizing a stronger form of order of functions on a set; (2) the existence of the integral can be characterized in terms of a suitably defined notion of g-content zero of the oscillation sets of f; (3) the existence of the integral is equivalent to the zero measure of the oscillation sets of f, giving another proof of this theorem for the usual Riemann-Stieltjes integral; (4) the additive property of integrals carries over in the form: if $S(B, f; B_i, g)$ exists with $\bigcup_i B_i=B$, B_i $(i=1,\cdots,m)$ being intervals, then $S(B, f; B, g)$ exists and $\sum_i S(B, f; B_i, g)=S(B, f; B, g)$. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Gomes, Ruy Luís. Riemann-Stieltjes integral in a locally compact space. Gaz. Mat., Lisboa **12**, no. 50, 97–100 (1951). (Portuguese)

Let E be a locally compact space, $\mathfrak{F}=\{f\}$ $(L=\{\varphi\})$ the set of all bounded (continuous) functions defined on E vanishing out of a compact set. An interval $V=[\varphi_1, \varphi_2]$ $(\varphi_1, \varphi_2\epsilon L)$ is the set of all $f\epsilon\mathfrak{F}$ such that $\varphi_1\leqq f\leqq\varphi_2$. Given a non-negative linear functional $F(\varphi)$ on L, the numbers

$$\bar{F}(f)=\inf_{V, V\supset f}[\sup_{\varphi\epsilon V} F(\varphi)] \quad\text{and}\quad \underline{F}(f)=\sup_{V\supset f}[\inf_{\varphi\epsilon V} F(\varphi)]$$

are called the upper and lower Riemann-Stieltjes integrals of the function $f\epsilon\mathfrak{F}$, and $f\epsilon\mathfrak{F}$ is said to be integrable if $\underline{F}(f)=\bar{F}(f)=F(f)$. In the case $E=(a, b)\subset R^1$, $F(f)$ coincides with the Stieltjes integral and $\bar{F}$ with the upper R-S integral defined by Pollard [Quart. J. Pure Appl. Math. **49**, 73–138 (1921)], and the properties of these integrals subsist for a general E. *M. Cotlar* (Chicago, Ill.).

Norris, M. J. Integrability of continuous functions. Amer. Math. Monthly **59**, 244–245 (1952).

The author bases the definition of a definite integral on a sequence of sub-divisions P_n where the end-points of P_{n-1} are contained among those of P_n and proves the existence

of the integral of a continuous function without using the idea of uniform continuity. Let P be a sub-division of $[a, b]$. Let

$$U(P)=\sum M_i(x_i-x_{i-1}),\quad L(P)=\sum m_i(x_i-x_{i-1}),$$

$$\overline{\int_a^b} f(x)dx=\sup U(P),\quad \underline{\int_a^b} f(x)dx=\inf L(P).$$

It is shown that for $a<c<b$

$$\overline{\int_a^c} f(x)dx+\overline{\int_c^b} f(x)dx=\overline{\int_a^b} f(x)dx,$$

and likewise for $\inf L(P)$. Set

$$z(x)=\overline{\int_a^x} f(x)dx-\underline{\int_a^x} f(x)dx$$

and prove that when $f(x)$ is continuous, $z'(x)=0$ at all points of (a, b). Hence z is constant, and since $z(a)=0$ the right side of (1) is zero for all x. Hence $\sup U(P)=\inf L(P)$.

R. L. Jeffery (Kingston, Ont.).

Fort, M. K., Jr. Some properties of continuous functions. Amer. Math. Monthly **59**, 372–375 (1952).

This is an exposition of certain continuity properties of complex functions, including Brouwer's fixed point theorem for the plane, from which is obtained a simple formulation of the proof of the fundamental theorem of algebra.

I. Niven (Eugene, Ore.).

Bononcini, Vittorio E. Su una estensione del campo di esistenza di una funzione continua in un insieme chiuso. Rivista Mat. Univ. Parma **2**, 365–374 (1951).

Let $f(P)$ be a real-valued function defined at each point P of a closed set C in an interval S of an (n-dimensional) Euclidean space E, and let it satisfy the (generalized) Lipschitz condition $|f(P)-f(P')|\leqq\omega(\overline{PP'})$ whenever P and P' are in C, the function $\omega(t)$ being concave (downwards) for $t\geqq 0$ and infinitesimal as $t\to 0$. Then a function $F(P)$ can be defined on S so as to coincide with $f(P)$ on C and to satisfy $|F(P)-F(P')|\leqq\omega(\overline{PP'})$ throughout S [cf. McShane, Bull. Amer. Math. Soc. **40**, 837–842 (1934); Whitney, Trans. Amer. Math. Soc. **36**, 63–89 (1934)]. The author constructs, by linear interpolation from McShane's function, a function $F_0(P)$ coinciding with $f(P)$ on C, quasi-linear on $S-C$ (that is, linear on each triangle of some subdivision of $S-C$, the set of vertices being dense only on C), and satisfying $|F_0(P)-F_0(P')|\leqq 170\omega(\overline{PP'})$ throughout S, which is taken to be two-dimensional for simplicity. He also obtains a similar result by slight modification of a construction used by Lebesgue [Rend. Circ. Mat. Palermo **24**, 371–402 (1907)] in obtaining a merely continuous extension of $f(P)$.

H. P. Mulholland (Birmingham).

Viola, Tullio. Sur la possibilité de compléter la définition d'une fonction donnée sur un domaine ouvert, par tendance à la limite vers la frontière du domaine. C. R. Acad. Sci. Paris **234**, 2513–2515 (1952).

This note states sufficient conditions in order that a real function defined on the interior of a domain shall have a unique limit almost everywhere on the boundary of the domain, which is also Lebesgue-integrable.

L. M. Graves (Chicago, Ill.).

Viola, Tullio. Domaines réguliers et domaines normaux. C. R. Acad. Sci. Paris **235**, 10–12 (1952).

A comparison of "regular domains" and "normal domains", with three examples in the plane.

L. M. Graves (Chicago, Ill.).

Zolotarëv, G. N. A sufficient condition for linear dependence of functions of one variable. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 201–205 (1952). (Russian)

Let W be the Wronskian matrix of n functions whose $n-1$ first derivatives are continuous and let W' be the matrix of the first $n-1$ rows of W. Then if W and W' have a same rank >0, the n functions are linearly dependent.

L. C. Young (Madison, Wis.).

Poplavskaya, G. Ya. On the equivalence of various definitions of area of a continuous surface. Mat. Sbornik N.S. **30(72)**, 651–668 (1952). (Russian)

The paper, which limits itself to surfaces of the type $z=f(x, y)$, presents in more detail the results of a previous note [Doklady Akad. Nauk SSSR **77**, 21–24 (1951); these Rev. **12**, 687].

L. C. Young (Madison, Wis.).

Glebskiĭ, Yu. V. Convergence in area and convergence by functional. Mat. Sbornik N.S. **30(72)**, 529–542 (1952). (Russian)

Let F be a positive continuous function of the 3-vector x and the unit 3-vector y, and let (f, F) be its integral over the surface with the Dirichlet representation $x=f(u, v)$ on the unit square. Assuming that the positively homogeneous extension of F as function of y is strictly convex, and that f describes a sequence converging uniformly to an f_0 such that the corresponding Dirichlet integrals are uniformly bounded, the author proves that the relation $(f_n, F)\to(f_0, F)$ implies convergence in area $[(f_n, 1)\to(f_0, 1)]$.

L. C. Young (Madison, Wis.).

Caccioppoli, Renato. Misura e integrazione sugli insiemi dimensionalmente orientati. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 137–146 (1952).

[For part I see same Rend. (8) **12**, 3–11 (1952); these Rev. **13**, 830.] This second note deals "for the sake of brevity" mainly with the case $n=3$, $k=1$ of what the author claims to be integration on a k-dimensional variety in n-space. The author criticizes the work of Federer and others as "laborious in deduction and artificial in hypothesis". Since the author does not trouble to do more than sketch his own hypotheses and his own deductions, comparison is hardly possible.

L. C. Young (Madison, Wis.).

Caccioppoli, Renato. Misura e integrazione sulle varietà parametriche. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 219–227 (1952).

In this note, again the first of a series, the author begins to sketch a revised version of his early work on surface area and allied topics. He rectifies what he terms minor errors and omissions and complains that these alone are quoted. He states that his theory has many advantages over subsequent theories and that it desires to be independent of special topological investigations. This last statement can only lead one to doubt its depth, but, in any case, it is difficult, without more details, to assess the scope and generality of a theory presented so sketchily. The reviewer has the strong impression that a careful analysis of the notions introduced so light-heartedly would lead more deeply into the topological background than the author considers desirable.

L. C. Young (Madison, Wis.).

Tolstov, G. P. On partial derivatives. Amer. Math. Soc. Translation no. 69, 30 pp. (1952).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 425–446 (1949); these Rev. 11, 167.

Ilieff, Ljubomir. Beitrag zum Problem von D. Pompeiu. Acad. Républ. Pop. Roum. Bull. Sect. Sci. 30, 613–617 (1948).

This is a translation from the Bulgarian version [Annuaire Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) 44, 309–316 (1948); these Rev. 12, 248]. *M. O. Reade.*

Theory of Functions of Complex Variables

Eggleston, H. G., and Ursell, H. D. On the lightness and strong interiority of analytic functions. J. London Math. Soc. 27, 260–271 (1952).

The authors define an index $j(z_0, C)$ of a point z_0 of the complex plane relative to an oriented 1-cycle C whose carrying set C' does not contain z_0 as the integral multiple of 2π by which amp $(z-z_0)$ changes as C is traversed, i.e., as the integral over C of d[amp $(z-z_0)$] divided by 2π. Using this index, a series of lemmas is established from which follows the two fundamental topological properties of the mapping generated by a non-constant analytic function, namely, lightness and strong interiority. Thus the authors succeed in establishing these properties without using the usual sophisticated machinery of analysis, such as development in power series. The results are obtained for a broader class of functions the definition of which does not require differentiability everywhere; and indeed, in Part II of the paper, they are obtained for a so called "ring-class field" of functions satisfying a weak form of the maximum modulus principle but in which no differentiability is explicitly assumed. *G. T. Whyburn* (Charlottesville, Va.).

Heffter, Lothar. Zur Begründung der Funktionentheorie. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1951, 293–304 (1951).

The author simplifies some of the proofs, and makes some historical remarks, regarding his former [Arch. Math. 1, 77–79 (1948); these Rev. 10, 240] discussion of analyticity in complex variable theory in terms of the Cauchy and Morera theorems for oriented rectangles.

E. F. Beckenbach (Los Angeles, Calif.).

Vakselj, Antoine. Contribution à la géométrie d'une fonction analytique. C. R. Acad. Sci. Paris 234, 797–799 (1952).

Given an analytic function $f(z)$ which is regular at a point z_0, the author introduces three linear transformations depending on $f(z)$ and z_0 which give rise to formal relations reminiscent of those between the tangent, normal, and binormal of a curve in space. *Z. Nehari* (St. Louis, Mo.).

Orts Aracil, José M.ª. Variations of the mean quadratic value of the modulus of an analytic function on a region bounded by an ellipse. Mem. Real Acad. Ci. Art. Barcelona 30, 429–437 (1951). (Spanish)

For an analytic function of a complex variable,

$$f(z)=\sum_{n=0}^{\infty} a_n z^n,$$

we have the familiar formula

$$(*)\qquad \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\varphi})|^2 d\varphi=\sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

The author now establishes the following generalization of (*): If $f(z)$ is analytic in an ellipse with foci at $(-c, 0)$ and $(c, 0)$, so that

$$f(t)=\sum_{n=0}^{\infty} a_n\{[z+\sqrt{(z^2-c^2)}]^n+[z-\sqrt{(z^2-c^2)}]^n\},$$

then for $a^2-b^2=c^2$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(a \cos \varphi+ib \sin \varphi)|^2 d\varphi =\sum_{n=0}^{\infty} |a_n|^2[(a+b)^{2n}+(a-b)^{2n}].$$

Geometrical applications and results concerning polynomial approximations are given. *E. F. Beckenbach.*

Ullman, J. L. Hankel determinants whose elements are sections of a Taylor series. II. Duke Math. J. 19, 155–164 (1952).

[Für den ersten Teil siehe Duke Math. J. 18, 751–756 (1951); diese Rev. 13, 221.] Es sei $f(z)=\sum_{k=0}^{\infty} a_k z^{-k-1}$ in $|z|>R$ regulär, auf $|z|=\sigma$ $(\sigma\leqq R)$ regulär, in $|z|>\sigma$ meromorph und die Gesamtordnung der Pole dort $\leqq p-1$. Es sei ferner $s_n(z)=\sum_{k=0}^{n} a_k z^{-k-1}$ und $S_{n,p}(z)=\|s_{n+i+j-2}\|_1^p$ die Hankelsche Determinante mit dem Glied $s_{n+i+j-2}$ in der i-ten Zeile und der j-ten Spalte. Verf. beweist, dass die Nullstellen von f in $|z|>\sigma$ Häufungspunkte der Nullstellen von $S_{n,p}(z)$ für $n\rightarrow\infty$ sind und zwar so, dass in beliebig kleinen Umgebungen einer r-fachen Nullstellen von f für genügend grosse n sich genau r Nullstellen von $S_{n,p}$ befinden. Dies verallgemeinert ein Theorem von Hurwitz $(p=0)$. Überdies kann $S_{n,p}$ in der Umgebung eines Poles von f bei genügend grossem n nicht mehr verschwinden. Der Beweis stützt sich wesentlich auf die folgende vom Verf. gegebene Verallgemeinerung eines Satzes von Hadamard: Es sei

$$f(z)=Q(z)/P(z)+G(z),$$

wobei

$$P=\prod_{i=1}^{k}(z-\alpha_i),\quad \alpha_i\neq 0,\quad |\alpha_i|=\rho_i,\quad \rho_i\leqq\rho_{i+1},$$

$i=1, 2, \cdots, k-1$, Q ein Polynom vom Grad $<k$ und $Q(\alpha_i)\neq 0$ und $G(z)$ regulär ist in $|z|>\rho_k$. Mit $A_{n,k}=\|a_{n+i+j-2}\|_1^k$ gilt dann

$$|A_{n,k}-Q(\alpha_1)\cdots Q(\alpha_k)\alpha_1^n\cdots\alpha_k^n|\leqq M(\mu\rho_1\rho_2\cdots\rho_k)^n,$$

wo die Grössen M und $\mu<1$ nur von den α_i abhängig sind.

A. Pfluger (Zürich).

***Privalov, I. I. Graničnye svoĭstva analitičeskih funkciĭ. [Boundary properties of analytic functions].** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 336 pp. (1 plate)

This book of 336 pages deals with a small and most difficult section of the theory of complex variables to which the author has made very substantial contributions [cf., e.g., Lusin and Privaloff, Ann. Sci. École Norm. Sup. (3) 42, 143–191 (1925)]. The book begins with an introduction to real and complex variables. The heads of the following chapters and some paragraphs are: Ch. 1: Boundary behavior of harmonic functions, bounded analytic functions (harmonic functions representable by a Poisson integral, Fatou's theorem, Blaschke's functions); Ch. 2: Boundary

behavior of functions analytic in a circle (different classifications of such functions); Ch. 3: Boundary behavior of functions analytic in domains bounded by rectifiable curves (integrals of Cauchy type, conformal representation, Privaloff's lemma on integrals of the Cauchy-Stieltjes type); Ch. 4: Uniqueness theorems for analytic functions with given behavior on the boundary (harmonic measure, uniqueness theorem of Lusin-Privaloff (limits in sectors), examples of analytic functions zero in a nullset on the boundary, Plessner's theorem on meromorphic functions on the boundary, Lusin-Privaloff's example of an analytic function in the unit circle with zero limits along radii in a set of positive measure, uniqueness theorem of Lusin-Privaloff on analytic functions with radial limits).

František Wolf (Berkeley, Calif.).

Garnier, René. Sur le problème de Riemann-Hilbert. Compositio Math. 8, 185–204 (1951).

The Riemann-Hilbert problem consists in the following. Given in the complex plane a simple closed curve C, the interior and exterior regions being R^+ and R^-, respectively, and given a square matrix of order n, $A(z)=(a_{jk}(z))$, defined on C, to construct in R^+ and R^- analytic matrices $\Phi(z)$, $\Psi(z)$ satisfying $\Phi=A\Psi$ on C. The early solutions (under various hypotheses) were given by Hilbert and Plemelj. G. D. Birkhoff [Proc. Amer. Acad. Arts Sci. 49, 521–568 (1913)] solved the problem by successive approximations under the assumptions: C is analytic and is a sum of a finite number of arcs interior to each of which $a_{jk}(z)$ coincides with an analytic function, while at the points of subdivision the $a_{jk}(z)$ are indefinitely differentiable relative to C. The author extends the method of Birkhoff to the case: C analytic, without singularities, the $a_{jk}(z)$ are Lipschitz (or, in fact, Hölder). This extension is based on the following theorem. Let $\tau(z)$, with $|\tau(z)|=1$ on C, be analytic on C, let the integer p be non-negative and $g(z)$, ϵ Lip (on C), be analytic interior to C; then the function

$$F_{jk}(X)=\frac{1}{2\pi i}\int_C \tau^p(z)a_{jk}(z)g(z)\frac{dz}{z-X}$$

satisfies $\max|F_{jk}(X)|\leqq\eta\cdot\max|g(z)|$, where η (positive and independent of $g(z)$) tends to zero with $1/p$; the maxima are attained on C. The author's method involves various trigonometric approximations of $A(z)$.

W. J. Trjitzinsky.

Gahov, F. D. Singular cases of Riemann's boundary problem for systems of n pairs of functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 147–156 (1952). (Russian)

This paper supplies the proofs of the results announced in an earlier note [Doklady Akad. Nauk SSSR 80, 705–708; these Rev. 13, 545].

M. Golomb (Lafayette, Ind.).

Tumarkin, G. C. On conditions of convergence of the boundary values of a sequence of analytic functions. Doklady Akad. Nauk SSSR (N.S.) 83, 655–658 (1952). (Russian)

Let G be a region bounded by a rectifiable Jordan curve γ. Let there be given a sequence of functions $f_n(z)$ analytic in G and possessing boundary limits in angle $f_n(\zeta)$ almost everywhere on γ. The author obtains a number of conditions which insure the convergence of $f_n(\zeta)$. Thus, e.g., let there exist two functions $g_i(z)\geqq 0$ $(i=1, 2)$ defined and continuous interior to G possessing boundary limits $g_i(\zeta)$, $g_1(\zeta)\neq 0$ almost everywhere. If $g_1(z)\leqq|f_n(z)|\leqq g_2(z)$ $(n=1, 2, \cdots)$, $z\epsilon G$, then a necessary and sufficient condition for the convergence in measure of $f_n(\zeta)$ on a set e, $m(e)>0$, is that (a) the sequence $\{|f_n(\zeta)|\}$ converges in measure on the set e, (b) the sequence $\{f_n(z)\}$ converges uniformly interior to G. Various consequences of this are obtained.

P. Davis.

Walsh, J. L., and Nilson, E. N. Note on overconvergence in sequences of analytic functions. Proc. Amer. Math. Soc. 3, 442–443 (1952).

The authors apply the principle of the "exact harmonic majorant" [see J. L. Walsh, Duke Math. J. 13, 195–234 (1946); these Rev. 8, 201] to results which they had obtained in a paper on approximation in the sense of least pth powers [Trans. Amer. Math. Soc. 65, 239–258 (1949); these Rev. 10, 524]. Equations concerning the measure of approximation of the functions $f_n(z)$ of best approximation to $f(z)$ (f given, analytic in some region R_ρ) and the norms of the $f_n(z)$ were proved in this paper. They are now interpreted in the sense of the exact harmonic majorant. Hence in the case when for a subsequence $\{f_{n_k}(z)\}$ (or any other sequence) there are inequalities similar to, but stronger than, the above equations, results about overconvergence of the $f_{n_k}(z)$ on the boundary of R_ρ are deduced. Thus a theorem, concerning lacunary structure of $f(z)$, of the former paper is considerably improved, and further results are suggested.

H. Kober (Birmingham).

Franck, A. Analytic functions of bounded type. Amer. J. Math. 74, 410–422 (1952).

Caractérisation des fonctions analytiques de type borné (quotient de deux fonctions holomorphes bornées) dans le demi-plan D $(x>0)$, au moyen de l'intégrale

$$\frac{1}{R_n}\int\int_{C_n}\frac{\log^+|f(z)|}{|1+z|^2}\,dxdy$$

étendue à une suite de cercles C_n de rayon R_n, épuisant D. Critères analogues pour un domaine limité par une courbe de Jordan, obtenus en particularisant un point frontière et en faisant certaines hypothèses de régularité. Applications. (Résumé d'une thèse de doctorat, University of Minnesota, 1949.)

J. Lelong (Lille).

Kahane, Jean-Pierre. Extension du théorème de Carlson et applications. C. R. Acad. Sci. Paris 234, 2038–2040 (1952).

The author first states a generalization of a theorem of Fuchs [J. London Math. Soc. 21, 106–110 (1946); these Rev. 8, 371] which in turn generalizes a well known theorem of Carlson. The result is as follows: Let $0<\lambda_n\uparrow\infty$, $D(u)=u^{-1}\sum_{\lambda_n\leq r}1$, and let $k(u)$ be a bounded measurable function; then if (i) $f(z)$ is analytic for $x>0$, $f(\lambda_n)=0$, $\log|f(z)|<|z|k(|z|)$, and

$$\text{(ii)}\qquad \limsup_{r\to\infty}\int^r(\pi D(u)-k(u))u^{-1}du=\infty,$$

then $f(z)\equiv 0$. Conversely, if in (ii) lim sup $<\infty$, and if $k(u)$ is of a bounded variation, there exists a function $f(z)\not\equiv 0$ satisfying (i). The above result and a certain refinement of the converse are then applied. Among the theorems stated we mention a theorem which relates the growth of an almost periodic function near a zero to the density of its positive spectrum. (Both cannot be too small without the function vanishing identically.) This problem for Fourier series was first discussed by Mandelbrojt [Séries de Fourier . . . , Gauthier-Villars, Paris, 1935] and was taken up later by Le-

vine and Lifschetz [Mat. Sbornik N.S. 9(51), 693–711 (1941); these Rev. 3, 106], Levin [Doklady Akad. Nauk SSSR (N.S.) 70, 949–952 (1950); these Rev. 12, 22], Hirschman and Jenkins [Proc. Amer. Math. Soc. 1, 390–393 (1950); these Rev. 12, 94] and Kahane and Lalaguë [C. R. Acad. Sci. Paris 230, 2250–2252 (1950); these Rev. 12, 22]. Finally it is stated that the results can be applied to give a quick derivation of some uniqueness and approximation theorems of Mandelbrojt [Ann. Sci. École Norm. Sup. (3) 65, 101–138 (1948); these Rev. 10, 436] as well as the extensions given by the reviewer [C. R. Acad. Sci. Paris 228, 1835–1837 (1949); these Rev. 11, 17]. *S. Agmon.*

Miyatake, Osamu. On the distribution of zero points of a function which is related to Riemann's ξ-function. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 2, 39–59 (1951).

Verf. untersucht die Nullstellen der ganzen Funktionen

$$\text{(A)}\qquad \xi_i(z)=2\int_0^\infty \psi_i(t)\cos zt\,dt,\quad i=1,2,3,$$

mit

$$\psi_1=\sum_{n=1}^\infty n^4\exp\left(-n^2\pi(e^t+e^{-t})\right)$$

$$\psi_2=\sum{}' n^4\exp\left(-n^2\pi(e^{+t}+e^{-t})\right)$$

$$\psi_3=\sum_{n=1}^\infty n^3\exp\left(-n^2\pi(e^t+e^{-t})\right),$$

wo $\sum'$ über alle n erstreckt wird, die keinen Primfaktor $>N$ enthalten. Die ξ_i stehen mit der Riemannschen ξ-Funktion in gewisser Verwandtschaft [vgl. hiefür auch G. Pólya, Acta Math. 48, 305–317 (1926)]. Für ξ_1 wurde kein Resultat erzielt, während bei ξ_2 und ξ_3 die Nullstellen mit genügend grossem Betrag alle auf der reellen Achse liegen und für deren Anzahl im Streifen $0\leqq x\leqq T$ die asymptotische Formel

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(1)$$

besteht. *A. Pfluger* (Zürich).

Miyatake, Osamu. Note on Riemann's ξ-function. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 2, 61–70 (1951).

Verf. betrachtet die ganze Funktion $\xi_4(z)$ (vgl. den Ausdruck (A) im vorangehenden Referat) mit

$$\psi_4=\sum{}'(e^{t/4}+e^{-t/4})\exp\left(-n^2\pi(e^t+e^{-t})\right)-\tfrac{1}{2}(e^{t/4}+e^{-t/4})^{-1}$$

wo $\sum'$ wie bei ψ_2 im vorangehenden Referat zu verstehen ist. Die Nullstellen genügend grosser Beträge sind alle reell und für ihre Anzahl im Streifen $0\leqq x\leqq T$ gilt

$$N(T)=\frac{T}{\pi}\log\frac{T}{\pi}+O(T).$$

A. Pfluger (Zürich).

Bojoroff, Eftim E. Über die Verteilung der Nullstellen einer Klasse von Polynomen und ganzen Funktionen. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 46, 43–72 (1950). (Bulgarian. German summary)

The author proves: (I) Let $f(t)$ be a positive increasing function and $\varphi(t)$ a positive decreasing function in $0\leqq t\leqq a$, and set

$$\text{(1)}\qquad F(u)=\int_0^{a-u}\{f(u+v)\varphi(v)+\lambda\varphi(u+v)f(v)\}dv,$$

$$\text{(2)}\qquad \Phi(u)=\int_0^{a-u}\{\varphi(u+v)f(v)+\lambda f(u+v)(v)\}dv,$$

$$P(z)=\int_0^a\{F(t)p(z+t)+\Phi(t)p(z-t)\}dt,$$

$|\lambda|=1$. If $p(z)$ is a polynomial with all its zeros in $\alpha\leqq\Re(z)\leqq\beta$, then $P(z)$ has all its zeros in the same strip. Other results of this type are also obtained. (II) Let $\lambda=1$ in (1) and (2) and set $s(u)=F(u)=\Phi(u)$, then the entire function $\int_0^a s(t)\cos zt\,dt$ has only real zeros. Pólya [Math. Z. 2, 352–383 (1918)] proved a similar result under the hypothesis that $s(t)$ is increasing. This represents an extension of Pólya's theorem since with a suitable choice of $f(t)$ and $\varphi(t)$, $s(t)$ will be decreasing.

(III) A necessary and sufficient condition that the zeros of

$$f_p(z)=\int_0^a\{\varphi(t)p(z+t)+\psi(t)p(z-t)\}dt$$

lie in the strip $0\leqq\Re(z)\leqq\beta$ whenever the zeros of the polynomial $p(z)$ lie in this strip, is that the zeros of the polynomials

$$\int_0^a\{\varphi(t)(1+zt)^m+\psi(t)(1-zt)^m\}dt,\quad m=1,2,\cdots,$$

all lie on the imaginary axis. (IV) A second necessary and sufficient condition for (III) is that

$$\Phi(z)=\int_0^a\{\varphi(t)e^{izt}+\psi(t)e^{-izt}\}dt$$

is an entire function of type II in the sense of Pólya and Schur [J. Reine Angew. Math. 144, 89–113 (1914)].

A. W. Goodman (Lexington, Ky.).

Davis, Philip. On the applicability of linear differential operators of infinite order to functions of class $L^2(B)$. Amer. J. Math. 74, 475–491 (1952).

Let B be a given domain in the complex plane. $L^2(B)$ is the class of functions $f(z)$, single-valued and regular in B, for which $\iint_B|f(z)|^2dxdy<\infty$; introducing the inner product $(f,g)=\iint_B f\bar{g}\,dxdy$, $L^2(B)$ becomes a Hilbert space. Linear bounded functionals in $L^2(B)$ are related to linear differential operators of infinite order. Except where otherwise stated, B is assumed to be schlicht, bounded, simply-connected, and such that if $b=$ boundary B then the complement of $B\cup b$ is a single region whose boundary is also b. For B there is known to be a unique complete orthonormal set of polynomials $p_n(z)=\sum_{k=0}^n a_{nk}z^k$ [$\mathrm{Re}(a_{nn})\geqq 0$, $a_{nn}\neq 0$].

Let $L(d)$ be the linear operator $L(d)=\sum_0^\infty\beta_n d^n$ defined by $L(d)f(z)=\sum_0^\infty\beta_n f^{(n)}(0)$, and associate with it the formal series (*) $L(z)=\sum_0^\infty\beta_n z^n$. Let $f\varepsilon L^2(B)$ have the Fourier expansion $f(z)=\sum_0^\infty\alpha_n p_n(z)$, $\alpha_n=(f,p_n)$; then $L(d)$ is said to be applicable (B) to f if the series

$$\sum_{n=0}^\infty\alpha_n\sum_{k=0}^n\beta_k p_n^{(k)}(0)=\sum_{n=0}^\infty\alpha_n\sum_{k=0}^n a_{nk}\beta_k\cdot k!$$

converges. $L(d)$ is then applicable (B) to all functions of $L^2(B)$ if and only if (**) $\sum_{n=0}^\infty\left|\sum_{k=0}^n a_{nk}\beta_k\cdot k!\right|<\infty$, and in this case the operator is bounded. Let $E^2(B)$ be the class of functions $L(z)$ of (*) for which (**) holds. Then every bounded

linear functional on the space $L^2(B)$ can be represented as an operator $L(d)$ for which $L(s)\epsilon E^2(B)$. Function $g(s)\epsilon E^2(B)$ if and only if g has the representation

$$g(s)=\int\int_B e^{sw}\overline{f(w)}\,du\,dv=\mathfrak{L}_B(f),$$

where $w=u+iv$ and $f(s)\epsilon L^2(B)$. This $\mathfrak{L}_B$-transform can be inverted: If $g=\mathfrak{L}_B(f)$ then $f(s)=\sum_{n=0}^{\infty}(\overline{g(d)p_n(s)})p_n(s)$. Other inversion formulas are also given. It is concluded that $g(s)\equiv 0$ if and only if $f(s)\equiv 0$, so there is a one-to-one correspondence between f and g. It is shown that there exist bounded simply-connected domains that fail to satisfy at least one of the conditions stated earlier for domains B, for which the $\mathfrak{L}_B$-transform is not one-to-one.

Let $k(\theta)$ be the supporting function of $\hat{B}$, the closed convex hull of B; then every $g(s)\epsilon E^2(B)$ is an entire function of exponential type, whose indicator $h(\theta)$ satisfies $h(\theta)\leqq k(-\theta)$, and whose conjugate indicator diagram is contained in B. Necessary and sufficient conditions are found in order that $L(s)$ have a prescribed indicator. Application is made to the problem of completeness for the set $\{e^{\lambda_n s}\}$, one result being this: Let $\{\lambda_n\}$ be distinct complex numbers with $\lambda_n\to\infty$, let $n(x)$ be the number of λ's satisfying $|\lambda_n|\leqq x$, and let $c(\hat{B})$ be the circumference of the convex hull of B. If

$$\limsup_{r\to\infty}\frac{1}{r}\int_1^r\frac{n(x)}{x}dx>\frac{1}{2\pi}c(\hat{B})$$

then $\{e^{\lambda_n s}\}$ is complete in $L^2(B)$. *I. M. Sheffer.*

***Gel'fond, A. O. Linear differential equations of infinite order with constant coefficients and asymptotic periods of entire functions.** Trudy Mat. Inst. Steklov., v. 38, pp. 42–67. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The author considers in the complex domain equations of the form $L[F]\equiv\sum_{n=0}^{\infty}a_nF^{(n)}(s)=\Phi(s)$, where $\phi(t)=\sum a_nt^n$ is an entire function of exponential type σ, and obtains several novel results. If α is a root of $\phi(t)=0$ of multiplicity $s+1$ (or more), then $z^se^{\alpha z}$ is a solution of $L[F]=0$, and the author shows first that any solution $F(z)$ of the homogeneous equation, regular in a circle $|z|\leqq\sigma_0$, $\sigma_0>\sigma$, is representable by a uniformly convergent series of these functions in some neighborhood of the origin. In fact,

$$(2\pi i)^{-2}\int_{|\xi|=\sigma_1+2\rho}\frac{F(\xi)}{\xi-z}\int_{|t|=R}\frac{dt}{\phi(t)}\int_0^{\infty}\frac{\phi(x/(\xi-z))-\phi(t)}{x/(\xi-z)-t}e^{-x}dx\,d\xi,$$

$\sigma_0>\sigma_1>\sigma$, is a linear combination of functions $z^se^{\alpha z}$, and the difference between this and $F(z)$ can be estimated. If $F(z)$ is an entire function, something can be said about the rapidity of convergence of the approximating sums.

The author next considers the inhomogeneous equation $L[F]=\Phi(z)$, with the object of showing that there is a solution of roughly the same growth as $\Phi(z)$ when $\Phi(z)$ is an entire function of greater than exponential type. His result is expressed in terms of a regularized maximum modulus $\bar{M}(r)$, defined for $r=|z|>1$ by

$$\log\bar{M}(r)=r\max_{1\leq t\leq r}t^{-1}\log M(t).$$

where $M(t)$ is the maximum modulus. Then if $\Phi(z)$ is of greater than exponential type, $\epsilon>0$ and $\theta>1$, the equation $L[F]=\Phi(z)$ has a solution $F_0(z)$ such that

$$|F_0(z)|<c(\epsilon,\theta)[\bar{M}(\theta\rho)]^{(1+\epsilon)/\log\theta},\quad |z|\leqq\rho.$$

A function F with period ω satisfies the equation

$$L[F]=F(z+\omega)-F(z)=0,$$

and the corresponding $\phi(z)$ is $e^{\omega z}-1$. Hence a generalization of the fact that a nonconstant entire function cannot have two independent periods is the theorem, which the author proves, that if $F(z)$ satisfies $L_1[F]=L_2[F]=0$, with $\phi_1(t)$ and $\phi_2(t)$ having no zeros in common, except perhaps the origin, then $F(z)$ must be a polynomial, of degree not exceeding one less than the multiplicity of the zero at the origin which has smaller multiplicity.

Finally the author generalizes Whittaker's definition of an asymptotic period [Interpolatory function theory, Cambridge, 1935] by requiring that $F(z+\omega)-F(z)$ is of lower order than $F(z)$ in a generalized sense. He calls $F(z)$ of greater order than $F_1(z)$ if there are positive increasing functions u and w, such that $u(2x)/u(x)\to 1$ and for some $\theta>1$

$$\limsup u[\log\bar{M}(r)]/w(r)>\limsup u[\log\bar{M}_1(\theta r)]/w(r),$$

$\bar{M}$ being defined as above. (If $u(x)$ and $w(x)$ are both $\log x$ we have the usual definition of order and asymptotic period.) Then if an entire function has any asymptotic periods in the generalized sense, they lie on a single straight line, they form a set of measure zero (these properties were proved by Whittaker with the old definition), and the ratio of two asymptotic periods is either a rational or a transcendental number.

R. P. Boas, Jr. (Evanston, Ill.).

Offord, A. C. Some remarks on Fréchet's space of integral functions. Proc. London Math. Soc. (3) 2, 60–68 (1952).

Eine Folge ganzer Funktionen nennt Verf. "konvergent", wenn sie in jedem beschränkten Bereich der komplexen Ebene gleichmässig konvergiert. Dadurch wird die Menge der ganzen Funktionen zu einem topologischen Raum R. Verf. zeigt von gewissen Eigenschaften, dass sie allen ganzen Funktionen bis auf eine Menge der Kategorie 1 zukommen. Z.B.:1. Es gibt in R eine Menge K der Kategorie 1, sodass jede ganze Funktion aus $R-K$ in jedem Parallelstreifen jeden komplexen Wert unendlich oft annimmt. 2. Ist $\{F_n\}$ eine Folge im Winkelraum $|\arg s|\leqq\frac{1}{2}\pi-\epsilon$ gelegener beschränkter und abgeschlossener Punktmengen F_n, sodass die Entfernung des Nullpunktes von F_n mit n gegen Unendlich strebt, so gilt für alle $f\epsilon R$ mit Ausnahme einer Menge der Kategorie 1

$$\liminf_{n\to\infty}\{\sup_{s\epsilon F_n}|f(s)|\}=\sigma.$$

Das letzte Beispiel zeigt, dass die von Littlewood und Offord für "fast alle" ganzen Funktionen erhaltenen Resultate [Ann. of Math. (2) 49, 885–952 (1948); diese Rev. 10, 692] nicht in $R-K$ gelten können. *A. Pfluger.*

Kasch, Friedrich. Über die eindeutige Primelementzerlegung. Norsk Mat. Tidsskr. 34, 10–12 (1952).

A short, simple proof of the proposition: (A) Factorization is unique in R_n, the ring of entire power series in the indeterminates $z_1,\cdots,z_n$ over the complex number field. In contradistinction to the usual proof, no use is made of the theorem: (B) If the integral domain I has unique factorization, so does $I[z]$. In fact, in the author's proof (B) is a consequence of (A).

J. Lehner (Philadelphia, Pa.).

Milloux, Henri. Sur les directions de Borel des fonctions entières, de leurs dérivées et de leurs intégrales. J. Analyse Math. 1, 244–330 (1951). (French. Hebrew summary)

A direction V is said to be a Borel direction of an entire function $f(z)$ of order $\rho>0$ if every angle bisected by V contains so many zeros of $f(z)-a$ that these zeros have their exponent of convergence equal to ρ except for at most one value of a. This paper contains the proofs of results announced earlier [C. R. Acad. Sci. Paris 231, 402–403 (1950); 232, 296–297 (1951); these Rev. 12, 170, 399]. The main result is that every entire function of finite positive order has at least one Borel direction in common with all its successive derivatives and integrals. The result is given in a more precise form in one direction: in this case $0<\rho<\infty$, no Borel directions are ever lost by (indefinite) integration. The other half of Valiron's problem, namely if Borel directions may disappear by differentiation, remains unsolved. For further details see the reviews mentioned above. [In line 8 of the first review, read "circles of repletion" instead of "circles".] *J. Korevaar* (Delft).

Goluzin, G. M. On the parametric representation of functions univalent in a ring. Mat. Sbornik N.S. 29(71) 469–476 (1951). (Russian)

For the real number q, $0<q<1$, let R_q denote the annulus $1<|z|=|x+iy|<p$, where $p=1/q$, and let $\bar{R}_q$ denote the closure of R_q. In this paper the author obtains a (parametric) representation of functions $f(z)$ which are analytic and univalent in R_q. The results overlap with those due to Komatu [Proc. Phys.-Math. Soc. Japan (3) 25, 1–42 (1943); these Rev. 7, 514]; but the present author's method is simpler and neater and does not make direct appeals to certain special functions.

First the author proves the following basic lemma. Let $f(z)$ be analytic in R_q, with Re $[f(z)]=$constant$=A$ on $|z|=1$. Then for $z\varepsilon R_q$,

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}[f(z')]K_q(z/z')d\theta+iB, \quad z'=pe^{i\theta},$$

where B is a real constant, where

$$K_q(\zeta)=\frac{1+\zeta}{1-\zeta}+\sum_{k=1}^{\infty}\left(\frac{1+q^{2k}\zeta}{1-q^{2k}\zeta}-\frac{\zeta+q^{2k}}{\zeta-q^{2k}}\right),$$

and where $(2\pi)^{-1}\int_0^{2\pi}\operatorname{Re}[f(z')]d\theta=A$. The proof of the lemma is obtained by a judicious use of the Laurent expansion for $f(z)$ and the Cauchy integral formulas. The function $K_q(\zeta)$ is shown to have the following properties: (i) it is analytic for $0<|\zeta|<\infty$ except for certain simple poles; (ii) it is positive for $q<|\zeta|<1$; (iii) the maximum (minimum) for Re $[K_q(\zeta)]$ on $|\zeta|=r$ is obtained for $\zeta=r$ $(\zeta=-r)$.

Now let $f(z)$ be analytic and univalent in R_{q_0}, and let $f(z)$ map this ring onto B_0, the domain $|w|>1$ with a slit L running out to $w=\infty$. Let $f(z)$ be normalized so that $|w|=1$ is the image of $|z|=1$, and $f(1)=1$. If the slit is shortened, by moving the finite endpoint along L, then one obtains a family of domains B_t, $0\leqq t\leqq\infty$, such that (i) B_∞ is $|w|>1$, (ii) $t_1\neq t_2$ implies $B_{t_1}\neq B_{t_2}$, (iii) $t_1<t_2$ implies $B_{t_1}\subset B_{t_2}$. Now let $g(z,t)$ be the univalent function that maps R_{q_t} onto B_t, such that $|w|=1$ is the image of $|z|=1$, and $g(1,t)=1$. Since $p_t=1/q_t$ is unique and is a continuous, strictly increasing function of t, we may choose the parameter t so that $p_t=q_0e^t$.

Now the function $w=f(z,t)=g^{-1}(g(z,0),t)$, $0\leqq t<\infty$, maps R_{q_0} onto R_{q_t} which has a slit terminating on $|w|=p_t$; moreover, $|z|=1$ is mapped onto $|w|=1$, and $f(1,t)=1$. Then, by introducing some auxiliary functions, the author derives the following Komatu-like differential equation:

$$(*) \qquad \frac{\partial}{\partial t}\log f(z,t)=-K_{q_t}(k\cdot q_t\cdot f(z,t))+K_{q_t}(kq_t),$$

where $k=k(t)$ is a continuous function with $|k(t)|=1$, $0\leqq t<\infty$. Hence every domain B, which is $|w|>1$ with a slit running out to $w=\infty$, has associated with it a continuous function $k(t)$, $0\leqq t<\infty$, $|k(t)|=1$, such that if $f(z)$, with $f(1)=1$, maps R_q in a univalent manner, onto B, then $f(z)$ has the representation

$$f(z)=\lim_{t\to\infty} f(z,t),$$

where $f(z,t)$ satisfies $(*)$ with $f(z,0)=z$. The author obtains a converse result. If $k(t)$ is continuous and if $|k(t)|=1$, $0\leqq t<\infty$, if $f(z,t)$ satisfies $(*)$, with $f(z,0)=z$, and if $\lim_{t\to\infty} f(z,t)=g(z)$ exists, then $g(z)$ is analytic and univalent in R_q, and $g(z)$ maps R_q onto some B above. *M. O. Reade* (Ann Arbor, Mich.).

Tamura, Jirô. Analytic representations of arbitrary automorphic functions. I. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 1, 11–14 (1951).

The author derives what is essentially a Mittag-Leffler representation of an arbitrary function automorphic on a Fuchsian group Γ (with unit circle as principal circle) in terms of Poincaré θ-series. However, he has apparently failed to take into account the possibility that the fundamental region of Γ may have vertices on the unit circle. His representation is not valid for a function having a polar singularity at such a vertex. *J. Lehner.*

Lohwater, A. J., and Piranian, G. Linear accessibility of boundary points of a Jordan region. Comment. Math. Helv. 25, 173–180 (1951).

Let R be a region in the plane and P a point of its boundary. The point P is said to be linearly accessible if there exists a segment PQ lying wholly in R, except for the point P. The authors construct a region bounded by a closed Jordan curve with the property that if the region is mapped conformally on the interior of a circle, the set of all linearly accessible boundary points is mapped on a set of points of the circumference of the circle of Lebesgue measure zero. *W. Seidel* (Rochester, N. Y.).

Komatu, Yûsaku. Mittlere Verzerrungen bei konformer Abbildung eines aufgeschlitzten Streifens. Kōdai Math. Sem. Rep. 1952, 1–4 (1952).

The author considers conformal mappings of a multiply-connected domain D onto canonical domains consisting of the strip $|\operatorname{Im}\{w\}|<\frac{1}{2}\pi$ which has been furnished with a number of horizontal or vertical slits. For greater convenience, D is also assumed to be confined between the lines $\operatorname{Im}\{z\}=\pm\frac{1}{2}\pi$ which form part of its boundary, and the mappings are made unique up to additive real constants by letting $\operatorname{Im}\{z\}=\pm\frac{1}{2}\pi$ and $\operatorname{Im}\{w\}=\pm\frac{1}{2}\pi$ correspond to each other. These mappings give rise to two domain constants whose dependence on D is investigated. *Z. Nehari.*

af Hällström, Gunnar. On the conformal mapping of incision domains. Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 13, 13 pp. (1952).

The author investigates the boundary correspondence in conformal mappings of the unit circle onto two types of

domains: (a) circles with an infinity of radial slits ending on the circumference; (b) rectangles with an infinity of slits parallel to one side and ending on the boundary. A number of criteria are given. *Z. Nehari* (St. Louis, Mo.).

Fichera, Gaetano. Sui teoremi d'esistenza della teoria del potenziale e della rappresentazione conforme. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 356–360 (1951).

Fichera, Gaetano. Sui teoremi d'esistenza della teoria del potenziale e della rappresentazione conforme. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 452–457 (1951).

These two notes are a curious mixture of mathematics and polemical remarks which it is not easy to disentangle. The first note opens with a general (verbal, not mathematical) attack on the kernel function method as applied to the solution of partial differential equations. The author expresses his belief that the kernel function method can neither be used for the proof of existence theorems nor for the numerical solution of boundary value problems, a belief for which no other authority but his word is given.

He then engages in criticizing a paper of Garabedian and Schiffer [Ann. of Math. **52**, 164–187 (1950); these Rev. **12**, 89; this paper will be referred to as G.-S.] which accomplishes precisely what he has just declared seems to him impossible, namely, the derivation by means of the kernel function method of a new set of existence proofs for the fundamental domain functions of potential theory and conformal mapping. (A paper by O. Lehto [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **59** (1949); these Rev. **11**, 170] in which the existence of the parallel slit mappings is proved in a similar way seems to have escaped both his notice and his criticism.) In view of the importance of this type of existence proof, it seems to the reviewer that a detailed examination of the facts is necessary.

In G.-S. there appears a function $H(x, y)$ which is harmonic in the closure of the unit circle except at the point $z=1$ ($z=x+iy$) and for which

$$\text{(1)} \qquad \iint_{|z|<1} (\operatorname{grad} H)^2\, dxdy < \infty.$$

The author's main criticism is then directed at the proof of G.-S. that

$$\text{(2)} \qquad \iint_{|z|<1} \left(\operatorname{grad} H, \operatorname{grad} \log \left|\frac{z-r^{-1}}{z-r}\right|\right) dxdy = 0$$

if $0<r<1$. G.-S. indeed make here a remark which is not correct but, as will be shown presently, this remark is completely irrelevant for the proof. The remark in question states that the integral in (2) may be evaluated by Green's formula since H is regular on $|z|=1$ except at $z=1$, and since, because of (1), H has a gradient which grows at most like $(1-|z|)^{-1}$ as $z\to 1$. Now it is quite evident that this growth can ruin the integrability of $\partial H/\partial n$ on $|z|=1$, even without the example the author constructs for this purpose. The author, incidentally, also doubts the assertion regarding the growth of grad H, indicating his misgivings by an exclamation point and a question mark enclosed in parentheses. His doubts, however, are unfounded, as one can easily confirm by applying to the identity

$$u(\zeta) = \iint_{|z|<1} (\operatorname{grad} k(z, \zeta), \operatorname{grad} u(z))\, dxdy$$

($k(z, \zeta)$ being the harmonic kernel function of the unit circle and $u(z)$ a harmonic function with a square-integrable gradient) the Schwarz inequality and using the explicit form of $k(z, \zeta)$.

But all this is unnecessary in order to prove (2). If C_ρ is the curve for which $|z-r| = \rho|1-rz|$ $(r<\rho<1)$, (2) can be replaced by $\lim_{\rho\to 1} A_\rho = 0$, where A_ρ is the same integral as in (2), except that $|z|<1$ is replaced by the interior of C_ρ. One may now evaluate A_ρ by Green's formula if a small circle about $z=r$ is deleted in the usual manner. We have

$$A_\rho = \int_{C_\rho} \frac{\partial H}{\partial n}\left[\log\left|\frac{1-rz}{r-z}\right| - \log r\right] ds = -\log(r\rho)\int_{C_\rho}\frac{\partial H}{\partial n} ds = 0,$$

and the contribution of the small circle vanishes. Thus $\lim_{\rho\to 1} A_\rho = 0$, and (2) is proved.

It thus appears that the reasoning in sections 2 and 3 of G.-S. is valid and that the author's criticism is unjustified. It should also be pointed out that in sections 4 and 5 of their paper G.-S. give an entirely different set of existence proofs, based on the use of a boundary norm and on properties of functions of bounded variation, respectively. These two sections do not come in for criticism. Since, however, these existence theorems contain those of sections 2 and 3 and since the proof of section 4 employs a kernel function, it is somewhat surprising that the author considers his disbelief in the proofs of sections 2 and 3 as sufficient justification for a sweeping condemnation of all existence proofs based on kernel functions.

It seems to the reviewer that in the heat of the argument the author fails to do justice to himself. In the second note he gives a solution of the Dirichlet problem which is analogous to an existence proof for a boundary-value problem in the theory of elasticity published by him in 1948 [same Rend. (8) **5**, 319–324 (1948); these Rev. **10**, 533], and which uses practically the same fundamental idea as the proof in section 3 of G.-S. The credit for having been the first to arrive at this new type of existence proof, which proceeds by first constructing a formal solution and then investigating its boundary behavior, thus definitely belongs to the author. Evidently, G.-S. were not aware of this work at the time they wrote their paper. That G.-S. use the kernel function while the author does not is but a superficial difference. The author starts from a complete set of harmonic functions $\{u_n(z)\}$ which are orthonormalized in the sense of the Dirichlet norm and he computes by means of Green's formula the corresponding Fourier coefficients $\{a_n\}$ associated with a function having given boundary values. He then shows—and this is the crux of the proof—that the harmonic function $\sum_n a_n u_n(z)$ indeed has the desired boundary values. G.-S. consider instead the harmonic kernel function, i.e. $\sum_n u_n(z)u_n(\zeta)$, and investigate its boundary behavior. The use of the special set of coefficients $\{u_n(\zeta)\}$ results in some added manipulative elegance, but otherwise there is no appreciable difference between the two versions; once the boundary behavior of the kernel is known, the solution of the general boundary-value problem is an immediate consequence. *Z. Nehari* (St. Louis, Mo.).

Sagawa, Akira. A note on a Riemann surface with null boundary. Tôhoku Math. J. (2) **3**, 273–276 (1951).

Verf. gibt eine Verallgemeinerung des Maximumprinzips an für beschränkte harmonische Funktionen auf einem nichtkompakten Gebiet einer nullberandeten Riemannschen Fläche [vgl. hiezu R. Nevanlinna, Ann. Acad. Sci. Fennicae. Ser. AI. Math.-Phys. no. **1** (1941); diese Rev. **7**, 427; M.

Ohtsuka, Nagoya Math. J. 3, 91–137 (1951); diese Rev. 13, 642; und bezüglich der Beweismethode, L. Sario, C. R. Acad. Sci. Paris 230, 269–271 (1950); diese Rev. 11, 342]. Die übrigen Resultate sind in dem für nullberandete Flächen gültigen Gross'schen Satz enthalten, der von K. Noshiro [Nagoya Math. J. 3, 73–79 (1951); diese Rev. 13, 833] bewiesen wurde, was Verf. bemerkt hat.

A. Pfluger (Zürich).

Rothstein, Wolfgang. Über die Fortsetzung von Verteilungen meromorpher Ortsfunktionen im R_6. Math. Ann. 124, 303–308 (1952).

In einer früheren Arbeit [Math. Ann. 121, 340–355 (1950); diese Rev. 11, 652] bewies Verf. gewisse Aussagen über die Fortsetzbarkeit von Verteilungen regulärer Ortsfunktionen (Cousin'scher Verteilungen zweiter Art) im Raum R_{2n}, $n\geqq 3$. In der vorliegenden Arbeit wird gezeigt, dass die gleichen Aussagen im wesentlichen auch für Verteilungen meromorpher Ortsfunktionen (Cousin'scher Verteilungen erster Art) gelten. Es sei dabei an folgenden Begriff erinnert: Jedem Punkte P eines Bereiches $\mathfrak{B}$ sei eine Umgebung U_P und eine dort meromorphe Funktion g_P so zugeordnet, dass stets bei zwei gegebenen Ortsfunktionen g_P und g_Q die Differenz g_P-g_Q in $U_P\cap U_Q$ regulär sei; dann heisse die Gesamtheit der gegebenen Ortsfunktionen und zugehörigen Umgebungen eine in $\mathfrak{B}$ zulässige Verteilung erster Art. Verf. gibt zunächst die wörtliche Übertragung seiner Sätze I und III der ersten Arbeit; aus Satz III folgt unter anderm die Gültigkeit des Kontinuitätssatzes auch für meromorphe Verteilungen. Nach geeigneter Erweiterung des Begriffs der zulässigen Verteilung durch Zulassung ausserwesentlich singulärer Stellen auf einer fest gegebenen analytischen Ebene gilt auch Satz II, dass nämlich eine auf einer abgeschlossenen analytischen Ebene des abgeschlossenen Raumes R_{2n} überall zulässige Verteilung in den ganzen abgeschlossenen R_{2n} fortsetzbar ist. Die Sätze werden für den Raum R_6 ausgesprochen, gelten aber allgemein für den R_{2n}, $n\geqq 3$.

P. Thullen (Genf).

Lelong, Pierre. Propriétés métriques des variétés analytiques complexes définies par une équation. Ann. Sci. École Norm. Sup. (3) 67, 393–419 (1950).

Let $f(z)$ denote a function, holomorphic, of p complex variables $z_1, \cdots, z_p$. The author's main tool for proving theorems on analytic varieties is the consideration of $\lambda[\log|f|; M, R]$, the average of $\log|f|$ taken on the spherical shell of radius R and center M, as a function of M and R. Considered as a function of $\log R$ it proves to be convex. This furnishes a simple proof of Poincaré's theorem: The $\log|f(M)|$ is the sum of a harmonic function and of a potential $U(M)=\int \overline{MQ}^{-2p+2}d\mu(Q)$ where μ is a positive distribution of constant density on the variety $f=0$. Hence μ is a constant times the area of $f=0$. If $\sigma(R)$ denotes the area inside of a sphere $S(M, R)$ with center at M and radius R, $\sigma(R)=\tau_{2p-2}\nu(M, R)R^{2p-2}$, where $\nu(M, R)$, the "average degree" of $f=0$ in $B(M, R)$, proves to be a non-decreasing function of R. Here $\nu(M, 0)$ is the multiplicity of $f=0$ at M and τ_{2p-2} denotes the $(2p-2)$-dimensional volume of the unit sphere. If $p\geqq 2$, then $\lambda(M, R)$ has a derivative and $\nu(M, R)=d\lambda/d\log R$. From the above follows that $\sigma(R)$ is at least equal to the area of the tangent plane (or cone) to $f=0$ at M inside $B(M, R)$. An upper estimate of the number of simplexes covering $f=0$ is obtained. Many more estimates and relations are obtained which belong to the same circle of ideas.

František Wolf (Berkeley, Calif.).

Bochner, S. Algebraic and linear dependence of automorphic functions in several variables. J. Indian Math. Soc. (N.S.) 16, 1–6 (1952).

Soient V une variété analytique-complexe compacte, $\tilde{V}$ son revêtement universel, Γ le groupe fondamental de V. Un système S de facteurs d'automorphie est une application $\alpha\rightarrow\eta_\alpha$ de Γ dans le groupe multiplicatif des fonctions holomorphes $\neq 0$ dans $\tilde{V}$, telle que $\eta_\beta(P)\eta_\alpha(\beta\cdot P)=\eta_{\alpha\beta}(P)$ pour α, $\beta\in\Gamma$, $P\in\tilde{V}$. Une fonction holomorphe dans $\tilde{V}$ est automorphe relativement au système S si $f(\alpha\cdot P)=\eta_\alpha(P)f(P)$. L'auteur démontre les deux théorèmes: S étant donné, l'espace vectoriel des fonctions automorphes relativement à S est de dimension finie (th. 4); si k est la dimension complexe de V, $k+2$ fonctions automorphes relativement à S sont algébriquement liées (th. 2). Ces résultats généralisent des théorèmes classiques et sont démontrés par des méthodes élémentaires. Le th. 4 résulte de l'énoncé suivant (th. 3): soient S une variété analytique-complexe, S^0 un ouvert relativement compact de S, F un espace vectoriel de fonctions holomorphes dans S, tel qu'il existe $M>0$ avec $\sup_{P\in S}|f(P)|\leqq M\cdot\sup_{P\in S^0}|f(P)|$; alors F est de dimension finie. [Note du rapporteur: ceci résulte du fait que F, muni de la topologie de la convergence uniforme, est localement compact. Cette démonstration prouve aussi le th. 5 que l'auteur donne sans démonstration comme conséquence des méthodes d'un mémoire à paraître.] Le th. 2 résulte de l'énoncé suivant (th. 1): soient $k+2$ fonctions f_i holomorphes dans une variété analytique-complexe S de dimension complexe k, et S^0 un ouvert relativement compact de S; s'il existe $M>1$ tel que, pour tout entier n et tout polynôme Q de degré n à $k+2$ variables, $\sup_S|Q(f_i)|\leqq M^n\cdot\sup_{S^0}|Q(f_i)|$, alors les f_i sont algébriquement liées.

H. Cartan.

San Juan, Ricardo. Sur la somme des classes quasi analytiques. C. R. Acad. Sci. Paris 235, 118–119 (1952).

The author gives a short proof that two functions, one analytic and the other belonging to a quasianalytic class, are identical if they and all their derivatives coincide at a point. [The result is not new: cf. T. Bang, Copenhagen thesis, 1946; C. R. Dixième Congrès Math. Scandinaves 1946, pp. 249–254 (1947); these Rev. 8, 199, 448.]

R. P. Boas, Jr. (Evanston, Ill.).

Lammel, Ernesto. Analogue of Riemann's theorem on conformal representation in the theory of the functions corresponding to $(\partial^2/\partial x^2-\partial^2/\partial y^2)U(x, y)=0$. Univ. Nac. Tucumán. Revista A. 8, 49–69 (1951). (Spanish. German summary)

For the analogue of complex variable theory given by Faber [Deutsche Math. 6, 323–341 (1942); these Rev. 4, 246], based on the equation $\partial^2U/\partial x^2-\partial^2U/\partial y^2=0$ in place of the Laplace equation, and on $\partial u/\partial x=\partial v/\partial y$, $\partial u/\partial y=\partial v/\partial x$ in place of the Cauchy-Riemann equations, the author now gives an analogue of the Riemann mapping theorem: the mapping of a simply connected plane domain onto the interior of the unit circle is replaced by the mapping of a Jordan arc onto a segment of a hyperbola. An analogue of the Koebe deformation theorem also is given.

E. F. Beckenbach (Los Angeles, Calif.).

The result quoted in the review is equivalent to one given by the author [Mem. Real Acad. Ci. Madrid. Ser. Ci Exact. 2 (1942), p. 12; these Rev. 14, 543] and so ante-dates the (independent) cited work of Bang.

Theory of Series

Jackson, F. H. Application of diluted matrices to bounded sequences. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 173–180, 181–190 (1952).

In a previous paper [Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 308–314 (1951); these Rev. 13, 340] the author introduced the notion of a "constant column diluted matrix", i.e. of a matrix obtained from a Toeplitz matrix A by inserting a fixed number of columns of zeros immediately preceding each column of A. By letting the number of columns of zeros introduced immediately before the k-th column of A depend on k, he now obtains a "variable column diluted matrix"; and he obtains a "matrix of variable dilution" by diluting individual rows of A with zeros, without regard for the manner of dilution used in other rows. Regularity of matrices (in the sense of G. H. Hardy and American writers), as well as the property of being convergence-preserving, is invariant under column dilution. Variable dilution may create (but not destroy) regularity, and it may create or destroy the property of being convergence-preserving. Two matrices A and B are said to be absolutely equivalent for a sequence s provided $As-Bs$ is a null-sequence. The author finds necessary and sufficient conditions for a matrix A and a given diluted version of A to be absolutely equivalent throughout the space of bounded sequences, and he exhibits specimens that satisfy these conditions.

In Part II, the author gives conditions on a matrix A under which there exist matrices, obtained by dilution of A, which are consistent with A and stronger than A.

G. Piranian (Ann Arbor, Mich.).

Tolba, S. E. On transformations by T- and γ-matrices. Nederl. Akad. Wetensch. Proc. Ser. A 55. = Indagationes Math. 14, 130–141; corrigenda, 345 (1952).

The author establishes simple necessary and sufficient conditions on a regular Toeplitz matrix in order that it evaluate a given bounded sequence with precisely two (or precisely three) limit points to a prescribed value. He proves several theorems of which the following is typical: If s and σ are two sequences, and if either s is unbounded, or s is divergent and σ is bounded, then there exist regular Toeplitz matrices that transform s into σ. The author discusses conditions under which some of these Toeplitz matrices are real, or real and non-negative; and he obtains the analogous results for series-to-sequence transformations. In a corrigendum, the author refers to a paper [Ann. of Math. 32, 715–722 (1931)] in which Agnew proved many similar theorems.

G. Piranian (Ann Arbor, Mich.).

Cooke, Richard G. On T-matrices at least as efficient as (C, r) summability, and Fourier-effective methods of summation. J. London Math. Soc. 27, 328–337 (1952).

The author proves that a regular method $A=(a_{nk})$ contains (C, r), $r>0$, if and only if (i) $\sum k^r|\Delta^r a_{nk}|\leqq M$, $n=1, 2, \cdots$, (ii) $|a_{nk}|\leqq K(n)k^{-r}$ for $n, k=1, 2, \cdots$. For $r=1$ this was given by Orlicz [Tôhoku Math. J. 26, 233–237 (1926)], for $0<r<1$ by the reviewer [Math. Z. 51, 85–91 (1948); these Rev. 10, 31]; condition (ii) may be omitted for $0<r<1$. In all cases, the result follows from the "Abel's transformation formula of order r" due for $0<r<1$ to the reviewer and for $r>1$ to Bosanquet [Proc. London Math. Soc. (2) 50, 482–496 (1949); these Rev. 10, 368; condition (a) $\sum|a_n|<+\infty$ on p. 88 of the reviewer's paper is misprinted to (a′) $\sum a_n$ converges; the author's remarks show that (a) and (a′) are equivalent under the other hypotheses made]. The author shows also that for Abel's method (A, p), $p>0$, $(C, 1)\subset(A, p)$. This is contained in a result of the reviewer [Canadian J. Math. 3, 236–256 (1951); these Rev. 13, 27] which gives all summability functions of methods (A, λ_n).

G. G. Lorentz (Kingston, Ont.).

Jurkat, W. Über Rieszsche Mittel mit unstetigem Parameter. Math. Z. 55, 8–12 (1951).

A form of Riesz's mean value theorem is that: if $A(t)$ is bounded, $0<k\leqq 1$, and $0<x<y$, then there is a number ξ, $0\leqq\xi\leqq x$, such that

$$\left|\int_0^x (y-t)^{k-1}A(t)dt\right|\leqq\left|\int_0^\xi (\xi-t)^{k-1}A(t)dt\right|$$

[M. Riesz, Acta Litt. Sci. Szeged 1, 114–126 (1923); for $A(t)$ a step function, Riesz, C. R. Acad. Sci. Paris 152, 1651–1654 (1911), Hardy and Riesz, The general theory of Dirichlet's series, Cambridge, 1915]. Here the author proves that if $A(t)$ is a step function and x belongs to an increasing sequence $\{\lambda_n\}$, then ξ may be chosen from the sequence $\{\lambda_n\}$, but that this need not be so if, for example, $A(t)$ is linear. As an application he establishes the known equivalence of the limits of $\sum_{\lambda_n\leqq\omega}(1-\lambda_n/\omega)a_n$, $0<k\leqq 1$, as $\omega\to\infty$ (a) continuously or (b) through the sequence $\{\lambda_n\}$ [Riesz, Proc. London Math. Soc. (2) 22, 412–419 (1924)].

L. S. Bosanquet (London).

Peyerimhoff, Alexander. Konvergenz- und Summierbarkeitsfaktoren. Math. Z. 55, 23–54 (1951).

The author uses functional analysis to obtain some comprehensive theorems concerning convergence and summability factors for matrix methods of summation. In a short review it is only possible to give a sample of his results. If A and B are convergence-preserving sequence-sequence (FF = Folge-Folge) transformations, then the sequence $\{\epsilon_n\}$ is said to belong to $(A, B)_f$ if $A(s_n\epsilon_n)$ converges whenever $A(s_n)$ converges. The author proves that if A is a triangular matrix with non-vanishing diagonal elements (A is normal) and $\{\epsilon_n\}$ belongs to $(A, B)_f$, then there is a sequence α_n such that

$$(*)\qquad \epsilon_n=\sum_{r=n}^{\infty}\alpha_r a_{nr},\quad \sum|\alpha_r|<\infty.$$

Conversely, if A satisfies certain further restrictions (A has the property SAK = schwache Abschnittskonvergenz), then (*) is sufficient as well as necessary. In particular, the result may be applied to obtain a theorem of Lorentz: that $\sum s_n\epsilon_n$ converges whenever $\{s_n\}$ converges (C, r), $0<r\leqq 1$, if and only if $\sum n^r|\Delta^r\epsilon_n|<\infty$ and $\sum\epsilon_n$ converges [G. G. Lorentz, Math. Z. 51, 85–91 (1948); these Rev. 10, 31; see also R. G. Cooke, paper reviewed second above]. On the other hand the result does not include the corresponding theorem for $r>1$, since the necessary condition $\epsilon_n=O(1/n^r)$ is not included in (*) [see Cooke, loc. cit., where this case is shown to be a transcription of a result of the reviewer, Proc. London Math. Soc. (2) 50, 482–496; these Rev. 10, 368; see also Mazur, Math. Z. 28, 599–611 (1928)]. A further application gives a convergence-factor theorem of W. Jurkat [Math. Z. 54, 262–271 (1951); these Rev. 13, 340]. There are corresponding results for series-sequence (RF = Reihen-Folge) transformations.

L. S. Bosanquet (London).

Jurkat, W., und Peyerimhoff, A. Mittelwertsätze bei Matrix- und Integraltransformationen. Math. Z. 55, 92–108 (1951).

A discrete analogue of Riesz's mean-value theorem [see theorem in the review of Jurkat second above], with fractional sums in place of fractional integrals, was stated by Jacob and found independently by the reviewer [M. Jacob, Proc. London Math. Soc. (2) **26**, 470–492 (1927); Bosanquet, J. London Math. Soc. **16**, 146–148 (1941) and Proc. London Math. Soc. (2) **50**, 482–496 (1949); these Rev. **3**, 144; **10**, 368]. Here the authors obtain a general class of triangular matrices $(a_{\mu\nu})$ for which the inequality

$$\left|\sum_{\nu=0}^{n} a_{m\nu}s_\nu\right| \leqq \left|\sum_{\nu=0}^{n'} a_{n'\nu}s_\nu\right|$$

holds for some n', $0\leqq n'\leqq m<n$. This class of matrices is identical with one defined from a different point of view by Peyerimhoff, which is significant in the theory of convergence and summability factors [$(a_{\mu\nu})$ has the property SAK; see review of Peyerimhoff above].

Conditions are given for $\{\epsilon_\nu\}$ to be a convergence factor for series summable by Riesz's typical means. The fact that these conditions differ for the two methods defined by the Riesz mean $\sum_{n\leqq\omega}(1-n/\omega)^\kappa$, according as $\omega\to\infty$ (a) continuously or (b) through the sequence n, gives a new proof of the non-equivalence of these two methods [cf. G. H. Hardy, Divergent series, Oxford, 1949, p. 114; these Rev. **11**, 25]. *L. S. Bosanquet* (London).

Zeller, Karl. Abschnittskonvergenz in FK-Räumen. Math. Z. 55, 55–70 (1951).

An FK-space E is a linear metric space of complex sequences $x=\{x_n\}$ with a topology defined by a sequence of quasi-norms [Zeller, Math. Z. **53**, 463–487 (1951); these Rev. **12**, 604]. E is said to have one of the properties AK(Abschnittskonvergenz = "section-convergence") or SAK (weak section-convergence) or AD (section-density) if, respectively, the $x^{(r)}$ converge to x, or weakly converge to x, or some $y^{(r)}$ converge to x, where $x^{(r)}=(x_1, \cdots, x_r, 0, \cdots)$, $y^{(r)}$ are finite sequences. The author gives criteria for E to have some of these properties. In particular, the sets of x summable to 0 by a given matrix A are FK-spaces; they often possess properties of AK-type. The main interest of the introduced notions is in their applications to summability. These include convergence factors, iteration of matrices. An intersection $\bigcap_{i=1}^{\infty}E_i$ of FK-spaces with $E_i\supset E_{i+1}$ is, under certain assumptions concerning their AK-properties, not a Banach space. It follows (as conjectured by the reviewer) that an intersection of an infinite sequence of decreasing matrix methods of summation is usually not equivalent to a matrix method. If for a perfect method A, $x_n=O(d_n)$, $d_n\to\infty$, for each A-summable x, then even $x_n=o(d_n)$ for any such x. There are relations to papers of Peyerimhoff [see the second preceding review] and of Jurkat and Peyerimhoff [see the preceding review] on convergence factors and methods with a mean-value theorem. *G. G. Lorentz.*

Wilansky, Albert. Convergence fields of row-finite and row-infinite reversible matrices. Proc. Amer. Math. Soc. 3, 389–391 (1952).

By constructing specific examples, Erdös and the reviewer have proved [same Proc. **1**, 397–401 (1950); these Rev. **12**, 92] that there exists a row-infinite (row-finite) regular Toeplitz matrix whose convergence field is not contained in the convergence field of any row-finite (row-infinite) regular matrix. The author carries the investigation further, replacing the hypothesis of regularity by the hypothesis of reversibility. He shows that if the matrix A is reversible, there exists a matrix, consistent with A, each of whose rows contains only finitely many zero elements, and whose convergence field is identical with that of A; and that the convergence field of A is contained in the convergence field of a normal convergence-preserving (but co-null) matrix. The author also shows that every reversible matrix has a right inverse, and that the converse is false: the matrix $(a_{n,k})$ with $a_{n,n}=-1$, $a_{n,n+1}=2$, $a_{n,k}=0$ $(k\neq n, n+1)$ has a two-sided inverse; but it can not be reversible, since it transforms the sequence $\{2^{-n}\}$ into the sequence $\{0\}$.

G. Piranian (Ann Arbor, Mich.).

Heller, I. Contributions to the theory of divergent series. Pacific J. Math. 2, 153–177 (1952).

Die Reihe $\sum u_n$ sei analytisch, d.h. $f(z)=\sum u_n z^n$ sei an $z=0$ regulär. Zunächst wird untersucht, wann die RR-Transformation von $\sum u_n$ mit Hilfe der Matrix $C=(c_{kn})$ stets wieder eine analytische Reihe $\sum v_k$ liefert; notwendig und hinreichend dafür ist, dass zu jedem $\epsilon>0$ ein M_ϵ existiert, sodass $|c_{kn}|\leqq\epsilon^{n+1}M_\epsilon^{k-1}$ $(k, n=0, 1, \cdots)$ gilt. Weiter bezeichne $\mathfrak{F}$ ein Gebiet der z-Ebene, das mit Hilfe der Singularitäten von $f(z)$ ähnlich wie das Borelsche Summabilitätspolygon erzeugt werden kann, wenn man dort das charakteristische Gebiet $\Re(z)<1$ durch ein allgemeineres Gebiet $\mathfrak{G}$ ersetzt. Es wird dann die Frage untersucht, wann die Anwendung von $C=(c_{kn})$ auf $\sum u_n z^n$ $(z\epsilon\mathfrak{F})$ auf eine Reihe $\sum v_k(z)$ führt, die Pt-summierbar ist, d.h. wann $y=1$ innerer Punkt des Hauptsterns von $\phi(y)=\sum v_k(z)y^k$ ist. Ähnliche Betrachtungen für FF-Transformationen. Eine Auseinandersetzung mit den bekannten Resultaten über Reihentransformationen von K. Knopp [Acta Math. **47**, 313–335 (1926)] und F. Lösch [Math. Z. **30**, 725–753 (1929)] wäre wünschenswert gewesen. *D. Gaier.*

Agnew, Ralph Palmer. Integral transformations and Tauberian constants. Trans. Amer. Math. Soc. 72, 501–518 (1952).

Let $\phi(x)$ be measurable for $x\geqq0$, $-\frac{1}{2}<\phi(x)<1$, $\phi(0+)=1$, $\phi(\infty)=0$, let $q_0>0$ be such that $\phi(x)>\frac{1}{2}$ for $x<q_0$, $\phi(x)<\frac{1}{2}$ for $x>q_0$ and let

$$A(q)=\int_0^q \frac{1-\phi(x)}{x}dx+\int_q^\infty \frac{|\phi(x)|}{x}dx$$

be finite for $q>0$. The author proves the following Tauberian theorem of Hadwiger's type: If

$$F(t)=\int_0^\infty \phi(xt)u(x)dx$$

and $t(\alpha)\to0$, $T(\alpha)\to\infty$ for $\alpha\to\infty$ and

$$0<q_1=\liminf\,(tT)\leqq\limsup\,(tT)=q_2<\infty,$$

then $\limsup_{\alpha\to0}|F(t)-S(T)|\leqq A\limsup_{x\to\infty}|xu(x)|$, where $S(T)=\int_0^T u(x)dx$ and $A=\max A(q)$ in (q_1, q_2); the constant A is the best possible. This is a generalization of a result of Rajagopal [Comment. Math. Helv. **24**, 219–231 (1950); these Rev. **12**, 494], where $Tt=q_0$ was assumed and conditions on ϕ were heavier. Similar results for more general kernels $\phi(x, t)$ were given by Delange [Ann. Sci. École Norm. Sup. (3) **67**, 99–160 (1950); these Rev. **12**, 253]. The smallest possible choice of A for all possible choices of functions $t(\alpha)\to0$, $T(\alpha)\to\infty$, independent of $u(x)$, is A_ϕ. If t, T are allowed to depend on $u(x)$, the problem becomes more

involved, but the minimal value of A is still A_0 if $\phi(x) \geqq 0$. The paper contains also numerical values for q_0 and A_0 in the cases: $\phi(x) = e^{-x}$; $\phi(x) = (1-x)^r$ on $(0, 1)$, $\phi(x) = 0$ for $x \geqq 1$, $r > 0$; $\phi(x) = (1-x)^{-r}$, $r > 0$; $\phi(x) = x/(e^x - 1)$, which correspond to the Laplace, Riesz, Stieltjes, Lambert transformations, respectively. *G. G. Lorentz* (Toronto, Ont.).

Levin, V. I. A limit estimate of the accuracy of asymptotic expansions of a certain class of functions. Doklady Akad. Nauk SSSR (N.S.) 80, 13–16 (1951). (Russian)

The purpose of this paper is to estimate

$$r(t) = \inf_{N \geqq 0} \left| f(t) - \sum_{n=0}^{N} a_n t^{-\gamma - nr} \right|$$

where (*) $\sum_{n=0}^{\infty} a_n t^{-\gamma - nr}$ is an asymptotic series for $f(t)$ as $t \to +\infty$. Three theorems are stated without proof; we reproduce the main assertions of the least general theorem: Let $\varphi_\zeta(t)$ have the Laplace transform

$$\varphi_\zeta^*(p) = (p - p_0)^{\gamma - 1} \sum_{\mu=1}^{m} A_\mu \{(p - p_0)^r - \zeta^r\}^{-\mu}$$

with $A_m \neq 0$, where r is a positive integer, γ is a real number $< r$ and not an integer, p_0 and $\zeta \neq 0$ are fixed complex numbers and p a complex variable restricted to the slit plane $|\arg (p - p_0)| < \pi$, and assume that no poles of $\varphi_\zeta(p)$ are situated on the slit. Then the function

$$f_\zeta(t) = e^{-p_0 t} \left\{ \varphi_\zeta(t) - \sum_{l=0}^{r-1} [\operatorname{res} \varphi_\zeta^*(p) e^{pt}]_{p = p_0 + \zeta \exp(2l\pi i/r)} \right\}$$

is asymptotically representable for $t > 0$ by (an explicitly given series) (*), and as $t \to \infty$ we have $\liminf r(t)/\rho(t) = 1$, $\limsup r(t)/\rho(t) = e^r$, where

$$\rho(t) = (2/\pi)^{1/2} |A_m \zeta^{r - (r-1)m + \gamma - 3/2} \sin \gamma\pi| \times \{(m-1)! r^{m-1} |\zeta^r - (-1)^r |\zeta|^r|\}^{-1} t^{m - 1/2} e^{-|\zeta| t}.$$

A. Dvoretzky (Jerusalem).

Kahan, Théo, et Eckart, Gottfried. Exposé d'ensemble des développements asymptotiques en physique ondulatoire. Revue Sci. 87, 3–24 (1949).

This expository paper written in an easily readable style summarises the various techniques of obtaining asymptotic expansions which may be used in problems of, e.g., nuclear physics, electromagnetic and acoustic waves, and geophysics. At first an introduction is given into the basic theory of an asymptotic series of the form $\sum a_n x^{-n}$. The main part of the paper is devoted to the discussion of Laplace's method and the methods of steepest descent and stationary phase. Many examples are given.

H. A. Lauwerier (Amsterdam).

Stuloff, N. Über die Entwickelbarkeit von Funktionen in verallgemeinerte Dirichletsche Reihen. Math. Z. 53, 273–284 (1950).

The object of this paper is to find necessary and sufficient conditions that a function $f(x)$ of a real variable $x \geqq 0$ be the sum of a convergent Dirichlet series $\sum a_n e^{-\lambda_n x}$ with non-negative coefficients a_n. They are found to be (i) $f(x)$ continuous for $x \geqq 0$ and infinitely differentiable for $x > 0$, (ii) for all n, $(-1)^n f^{(n)}(x) \geqq 0$,

(iii) $\sum_{\Lambda} \lim_{n \to \infty} (-e/\Lambda)^n f^{(n)}(n/\Lambda) + \lim_{n \to \infty} f(n) = f(0)$.

From the first two conditions it follows that the limit under the summation sign is $\neq 0$ for a denumerable number of Λ's only, which prove to be the exponents λ_n's. The proof is conducted in the real and uses relation between $f(x)$ and $\Psi(x)$ in $f(x) = \int_0^1 u^x d\Psi(u)$. *František Wolf.*

Stuloff, Nicolaus. Absolut konvergente verallgemeinerte Dirichletsche Reihen im Reellen. Math. Z. 54, 329–338 (1951).

The author continues his investigations of the preceding paper. First he proves a uniqueness theorem: If $f(x)$ may be represented for $x \geqq 0$ by two convergent Dirichlet series $\sum a_n e^{-\lambda_n x}$, $\sum a_n' e^{-\lambda_n' x}$, of which the first is absolutely convergent, then $\lambda_n = \lambda_n'$, $a_n = a_n'$. Using a theorem due to F. Hausdorff [Math. Z. 16, 220–248 (1923)], the author proves the following main theorem. A continuous $f(x)$ defined in $x \geqq 0$, infinitely differentiable in $x > 0$, is the sum of an absolutely convergent Dirichlet series $\sum a_n e^{-\lambda_n x}$ if and only if

$$\lim_{n \to \infty} \sum_0^n \binom{n}{\nu} |\Delta^{n-\nu} f(\nu)| = \sum_{\Lambda} \lim_{n \to \infty} \left(-\frac{e}{\Lambda}\right)^n f^{(n)}\left(\frac{n}{\Lambda}\right)$$

and the limit in the first member exists as a finite number.

František Wolf (Berkeley, Calif.).

Fourier Series and Generalizations, Integral Transforms

Chak, A. M. On the convergence and summability-(C, 1) of an analogous conjugate Fourier series. Bull. Calcutta Math. Soc. 43, 113–118 (1951).

The series $-\frac{1}{\pi}\sum_{n=1}^{\infty} \int_0^{2\pi} f(t) \sin [n\pi \sin \frac{1}{2}(t-x)] dt$ exhibits the same behavior with regard to convergence and $(C, 1)$ summability as conjugate Fourier series. The analogous conjugate function, however, is given by the formula

$$\tilde{f}(x) = -\int_0^{2\pi} \frac{f(x+t) - f(x-t)}{2 \tan \frac{1}{2}t} \frac{dt}{(\pi^2 - t^2)^{1/2}}.$$

P. Civin (Eugene, Ore.).

du Plessis, N. The Cesàro summability of Laplace series. J. London Math. Soc. 27, 337–352 (1952).

The following theorem is established. Suppose $f(Q)$ is integrable over the unit sphere, with the coordinates (θ, φ) chosen so that P is the pole. Then if

$$(*) \quad F(\theta) - F(0) = \int_0^{2\pi} (f(\theta, \varphi) - f(0, \varphi)) d\varphi = o(\theta^{1/2 - k}), \quad -\tfrac{1}{2} < k < \tfrac{1}{2},$$

the Laplace series for $f(Q)$ is (C, k)-summable at P to the value $f(P)$. The theorem fails if in (*) the o is replaced by O.

P. Civin (Eugene, Ore.).

Burkill, J. C. Uniqueness theorems for trigonometric series and integrals. Proc. London Math. Soc. (3) 1, 163–169 (1951).

The author has expressed [same Proc. (3) 1, 46–57 (1951); these Rev. 13, 126] in Fourier form the coefficients of a trigonometric series which is known to converge everywhere. A sufficiently comprehensive concept of integration for this purpose proved to be the symmetrical Cesàro-Perron integral (SCP integral). The discussion covered the case of (C, k)-summability for $k \leqq \frac{1}{2}$. The case $k > \frac{1}{2}$ and Abel's summation is taken up in this paper. The author states or deduces a number of theorems on the uniqueness of a trigo-

nometric series originally given for Denjoy integrable sum-functions by Verblunsky [ibid. (2) 31, 370–386 (1930)] and F. Wolf [ibid. 45, 328–356 (1939); these Rev. 1, 225], and on the uniqueness of trigonometric integrals given under the same conditions by Offord [ibid. 42, 422–480 (1937)] and F. Wolf [Univ. California Publ. Math. 1, 159–228 (1947); these Rev. 9, 140]. Using the SCP integral the author can dispense with the condition of Denjoy integrability of the sum-function. *František Wolf.*

Zitarosa, Antonio. Alcune estensioni dei teoremi di Young-Hausdorff e di Paley. Giorn. Mat. Battaglini (4) 3(79), 209–224 (1950).

The well-known theorems of Young-Hausdorff and Paley, established first for trigonometric functions [cf., e.g., Zygmund, Trigonometric series, Warsaw, 1935, p. 189 ff.], and later by F. Riesz [op. cit., p. 200] for general orthonormal, uniformly bounded systems of functions $\{\varphi_n(x)\}$, are extended by the author to systems which are not necessarily uniformly bounded, but such that, for all $|\varphi_n(x)|\leqq\psi(x)\epsilon L^\delta$, $\delta>3$. The polynomials of Legendre form a system of the second but not of the first kind. *František Wolf.*

Newman, Jerome, and Rudin, Walter. Mean convergence of orthogonal series. Proc. Amer. Math. Soc. 3, 219–222 (1952).

Does the Legendre expansion of every function $f\epsilon L^p$ on $(-1, 1)$ converge in the mean of order p to f? H. Pollard has shown the answer to be yes if $4/3<p<4$, and no if $p>4$ or $1\leqq p<4/3$ [Trans. Amer. Math. Soc. 62, 387–403 (1947); these Rev. 9, 280]. The authors show that the answer is also negative in the limiting cases $p=4$ and $p=4/3$. A similar completion is given to Pollard's results on the weighted L^p convergence of series of Jacobi polynomials [Duke Math. J. 16, 189–191 (1949); these Rev. 10, 450] and to G. M. Wing's results on the weighted L^p convergence of Dini series [Amer. J. Math. 72, 792–808 (1950); these Rev. 12, 329]. *J. Korevaar* (Delft).

Graves, Ross E. A closure criterion for orthogonal functions. Canadian J. Math. 4, 198–203 (1952).

Let $\{\varphi_n\}$ $(n=0, 1, 2, \cdots)$ be orthonormal on (a, b). The author uses Parseval's theorem to show that $\{\varphi_n\}$ is complete in $L^2(a, b)$ if and only if for some c, $p(t)$ and $w(x)$,

$$\sum_{n=0}^{\infty}\int_a^b\left|\int_c^x p(t)\varphi_n(t)dt\right|^2 w(x)dx=\int_a^b\left|\int_c^x |p(t)|^2dt\right|w(x)dx.$$

Here $a\leqq c\leqq b$, while $p(t)$ and $w(x)$ have to satisfy some obvious conditions. This completeness criterion is essentially due to D. P. Dalzell [J. London Math. Soc. 20, 87–93, 213–218 (1945); these Rev. 7, 437] and is used to prove the completeness in $L^2(-\infty, \infty)$ of the Hermite functions

$$\varphi_n(x)=(\pi^{\frac{1}{2}}n!2^n)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}H_n(x).$$

J. Korevaar (Delft).

Orlicz, W. On a class of asymptotically divergent sequences of functions. Studia Math. 12, 286–307 (1951).

$\{f_n\}$ converges asymptotically in A to f, if for every $\epsilon>0$, $\lim_{n\to\infty}$ meas $\{x\,|\,|f_n(x)-f(x)|\geqq\epsilon\}=0$ which shall be denoted by lim as$_A f_n=f$; $\{f_n\}$ is asymptotically bounded in A, if $\lim\theta_n=0$ implies lim as$_A\theta_nf_n=0$. Rademacher has defined the periodic $\epsilon_1(t)$ to be 1 for $0<t<\frac{1}{2}$, -1 for $\frac{1}{2}<t<1$ and 0 for $t=0, \frac{1}{2}$. Further $\epsilon_n(t)=\epsilon_1(2^nt)$ and $\eta_n(t)=\frac{1}{2}-\frac{1}{2}\epsilon_n(t)$. The paper consists of a great wealth of results involving the above concepts. For the sake of illustration we quote here the following. Theorem 4: If lim as$_Af_n^{(i)}=f_n$ and $\sum_{n=1}^{\infty}[f_n^{(i)}(x)]^2<\infty$ for all i, $x\epsilon A$, then

$$F_i(t, x)=\sum_1^{\infty}\epsilon_n(t)f_n^{(i)}(x)$$

is such that $F_i(t, x)$ converges asymptotically in a set Q of the x-axis of positive measure, either for almost all t, or for almost no t. Theorem 8: $\{g_n(x)\}$ measurable in (a, b), $A=\{\alpha\,|\sum_1^{\infty}g_n^2(x)<\infty\}$, then there exists a T, meas $T=1$, such that for $t\epsilon T$, $\sum_1^{\infty}\epsilon_n(t)g_n(x)$ converges almost everywhere and if meas $A<b-a$, then the last series is not asymptotically bounded on every subset E of $(a, b)-A$ of positive measure and for $\alpha>0$ lim sup$_{m\to\infty}\int_E|\sum_1^m\epsilon_n(t)g_n(x)|^\alpha dx=\infty$. Theorem 11: let φ be absolutely continuous and of period l and $\int_0^l\varphi'^2dx<\infty$, $\alpha_n>0$, $\sum_1^{\infty}\alpha_n<\infty$, $\lim\beta_n=\infty$, and set

$$\Phi(x, t)=\sum_1^{\infty}\epsilon_n(t)\alpha_n\varphi(\beta_nx),\quad \Phi^*(x, t)=\sum_1^{\infty}\eta_n(t)\alpha_n\varphi(\beta_nx).$$

If $\sum(\alpha_n\beta_n)^2=\infty$, then Φ, Φ^* are for almost any t not approximately differentiable almost everywhere. If $\sum(\alpha_n\beta_n)^2<\infty$, then Φ is differentiable in the ordinary sense for almost any t almost everywhere. The same is true for Φ^* under a supplementary hypothesis that $\sum\alpha_n\varphi(\beta_nx)$ is differentiable almost everywhere. *František Wolf* (Berkeley, Calif.).

Genuys, François. Sur les fonctions presque périodiques dans une bande. C. R. Acad. Sci. Paris 234, 1939–1941 (1952).

Let $F(s)$ be analytic and bounded in a strip $|\mathrm{Re}\,s|<d$, and let $\{\lambda_n\}$ be a sequence of positive integers. It is assumed that $F(s)$ is represented asymptotically, towards one end of the strip, by a sequence of exponential sums

$$(*)\qquad \sum_{n\leqq m}\{a_n^{(m)}\exp(-\lambda_ns)+b_n^{(m)}\exp(\lambda_ns)\}.$$

Here "asymptotic representation" is meant in the sense considered by Mandelbrojt [Ann. Sci. École Norm. Sup. (3) 63, 351–378 (1947); these Rev. 9, 229, 735]. If the strip is sufficiently wide, and if the approximation by the sums (*) is sufficiently close, then $F(s)$ must be almost periodic in some strip $|\mathrm{Re}\,s|<\delta$, the Fourier exponents of $F(s)$ belonging to the sequences $\{\lambda_n\}$, $\{-\lambda_n\}$. The proof is based on Mandelbrojt's results [loc. cit.]; only an outline is given. *J. Korevaar* (Delft).

Peck, J. E. L. Almost periodic functions. Proc. Amer. Math. Soc. 3, 107–110 (1952).

Soit D un ensemble de représentations d'un groupe G, irréductibles normales, unitaires et non équivalentes, il existe un ensemble φ_γ de fonctions p. p. particulières, non négatives, à valeur moyenne unité, et ayant un développement de Fourier fini, dont l'indice γ parcourt un ensemble dirigé Γ [cf. G. Birkhoff, Ann. of Math. 38, 39–56 (1937)] et telles que, $f(x)$ désignant une fonction p. p. de $x\epsilon G$ à valeurs complexes et dont le développement de Fourier ne contient que des éléments de D, on ait: $\lim_{\gamma\epsilon\Gamma}\varphi_\gamma\times f=f$. Comme

$$\varphi_\gamma\times f=\sum_{D\epsilon\mathfrak{D}}r_D{}^\gamma s^D\sum_{\rho,\sigma=1}^{s^D}C_{D,\rho,\sigma}(f)D_{\rho,\sigma}\quad(0\leqq r_D{}^\gamma\leqq1),$$

on a un procédé général de sommation des séries de Fourier des fonctions p. p. dans G lorsque $\mathfrak{D}$ désigne l'ensemble de toutes les représentations de G irréductibles, normales, unitaires et non équivalentes.

Les notations sont celles de von Neumann [Trans. Amer. Math. Soc. 36, 445–492 (1934)], la représentation de $x \in G$ par D est la matrice carrée $D_{\rho,\sigma}(x)$ ($\rho, \sigma = 1, \cdots, s^D$) dont les éléments sont p. p.; $C_{D,\rho,\sigma}$, $f \times g$, sont définis par les moyennes:

$$C_{D,\rho,\sigma}(f) = M_x\{D_{\sigma,\rho}(x^{-1})f(x)\}, \quad f \times g = M_y\{f(xy^{-1})g(y)\}.$$

J. Favard (Paris).

Takano, Kinsaku. Certain Fourier transforms of distributions. Tôhoku Math. J. (2) 3, 306–315 (1951).

The author refers to a paper by Lukacs and Szász [Canadian J. Math. 3, 140–144 (1951); these Rev. 12, 823] where a necessary condition was given that the reciprocal of a polynomial without multiple roots must satisfy in order to be a characteristic function. The author shows that the proof can be extended to cover the case when the polynomial has multiple roots. He found his result independently; some further results concerning such polynomials are included. *O. Szász.*

Lukacs, Eugene. An essential property of the Fourier transforms of distribution functions. Proc. Amer. Math. Soc. 3, 508–510 (1952).

The author considers an integral transform

$$f(s) = \int_{-\infty}^{+\infty} K(s, t)\,dF(t)$$

of a distribution function $F(t)$, assumed to satisfy $F(-\infty) = 0$, $F(+\infty) = 1$, and observes that the Fourier transform (that is, the special case $K(s, t) \equiv e^{its}$) has the following three properties, which make it useful in the theory of probability. (I) The kernel $K(s, t)$ is a complex-valued function defined and bounded for all real s and t and B-measurable in t; (II) for two distributions, $f_1(s) \equiv f_2(s)$ if and only if $F_1(t) \equiv F_2(t)$; (III) if $F(x) \equiv F_1 * F_2$, that is $\int_{-\infty}^{+\infty} F_1(x-t)\,dF_2(t)$, then $f(s) = f_1(s)f_2(s)$. He shows that a kernel $K(s, t)$ satisfies (I), (II), and (III) if and only if it has the form $K(s, t) = e^{itA(s)}$, where $A(s)$ is a real-valued function assuming all values of a set dense in the real line and $A(s_1) \neq A(s_2)$ when $s_1 \neq s_2$.

H. P. Mulholland (Birmingham).

Erdélyi, A. On some functional transformations. Univ. e Politecnico Torino. Rend. Sem. Mat. 10, 217–234 (1951).

Let $g(x) = H_\nu f = \int_0^\infty J_\nu\{2(tx)^{1/2}\}f(t)\,dt$ be the Hankel transformation of order ν; let $\mu \neq \nu$ be given. Then there exists an operator T of fractional integration or differentiation such that $Tg = H_\mu(Tf)$. Thus the Hankel transform of order ν is reduced to that of order μ and, therefore for $\mu = \pm\frac{1}{2}$, to Fourier transforms. This result, and the theory of certain operators of fractional integration had been given by the author and the reviewer [Quart. J. Math., Oxford Ser. 11, 193–211, 212–221 (1940); these Rev. 2, 191, 192]. A related method is used here to reduce a class of generalised hypergeometric transforms, some of which have been dealt with in the literature [C. S. Meijer, Nederl. Akad. Wetensch., Proc. 43, 599–608, 702–711 (1940); 44, 727–737, 831–839 (1941); R. P. Boas, Bull. Amer. Math. Soc. 48, 286–294 (1942); Proc. Nat. Acad. Sci. U. S. A. 28, 21–24 (1942); these Rev. 2, 96; 3, 38, 109, 233], to the Laplace transform $g(x) = Lf = \int_0^\infty e^{-tx}f(t)\,dt$.

Let

$$1 \leqq p, q \leqq \infty; \quad 1/p + 1/q = 1; \quad m > 0;$$
$$\Re(\alpha) > 0; \quad \Re(\eta) > -1/q; \quad \Re(\zeta) > -1/p;$$

$$\mathfrak{H}_{m,\alpha,\eta} f = \frac{m}{\Gamma(\alpha)} x^{-\eta - m\alpha + m - 1} \int_0^x (x^m - u^m)^{\alpha - 1} u^\eta f(u)\,du,$$

and let $\mathfrak{K}_{m,\alpha,\zeta} f$ be a similar operator, but corresponding to the Weyl fractional integral. Both $\mathfrak{H}$ and $\mathfrak{K}$ are generalizations of operators discussed in the papers referred to above. By means of them two "reciprocal" kernels

$$h(x) = h(\alpha, \beta, \zeta, m; x) \quad \text{and} \quad k(x) = k(\alpha, \beta, \eta, m; x)$$

are defined which can be expressed in terms of generalised hypergeometric series for $m = 2, 3, \cdots$, of confluent hypergeometric functions for $m = 1$. The author deals with the operators $\mathfrak{S}f = \int_0^\infty h(tx)f(t)\,dt$, $\mathfrak{T}f = \int_0^\infty k(tx)f(t)\,dt$ both of which reduce to Lf when $\alpha = \beta = 0$, and proves: if $f(t) \in L_p(0, \infty)$ then (a) $\mathfrak{T}f = x^\beta L[x^\beta \mathfrak{H}_{m,\alpha,\eta} f]$, and (b) $\mathfrak{T}f = \mathfrak{K}_{m,\alpha,\eta}[x^\beta L(x^\beta f)]$. Similarly $\mathfrak{S}f = x^\beta L[x^\beta \mathfrak{K}_{m,\alpha,\zeta} f]$ and $\mathfrak{S}f = \mathfrak{H}_{m,\alpha,\zeta}[x^\beta L(x^\beta f)]$. Finally an inversion and representation theory of $g(x) = \mathfrak{T}f$, $\mathfrak{S}f$ is given, based on that of the Laplace transformation.

H. Kober (Birmingham).

San Juan, Ricardo. Characterizations of generalized Laplace transforms in the spaces L, L^r, R and U. Revista Mat. Hisp.-Amer. (4) 12, 41–62 (1952). (Spanish)

The author is concerned with characterizing from among the transformations $I(\psi, s) = \int_0^1 \psi(x)\alpha(s, x)\,dx$ those of the form $\mathfrak{L}(\psi, s) = \int_0^1 \psi(x)x^{\sigma(s)-1}\,dx$. The present paper continues the ideas of a previous paper by the author [Portugaliae Math. 10, 115–120 (1951); these Rev. 13, 551].

I. I. Hirschman, Jr. (St. Louis, Mo.).

Pilatovskiĭ, V. P. On the computation of the remainder term of the asymptotic expansion of a function given by its Laplace transform. Doklady Akad. Nauk SSSR (N.S.) 83, 649–650 (1952). (Russian)

If $F(s)$ is the Laplace transform of $f(t)$, then the application of Poisson's summation formula [Pilatovskiĭ, same vol., 197–200; these Rev. 13, 647] leads to the approximation

$$f(t) \approx \frac{\lambda}{\pi} \sum_k e^{(\sigma + ik\lambda)t} F(\sigma + ik\lambda)$$

in which $\lambda\pi < t$ and k runs through all odd integers. The author uses this, with $F(s) = O(s^{-n})$, $f(t) = O(t^n)$ to discuss the remainder in asymptotic expansions. [Reviewer's remark: The note contains neither a precise result nor a convincing proof.] *A. Erdélyi* (Pasadena, Calif.).

Cotte, Maurice. Sur une correspondance symbolique approchée. C. R. Acad. Sci. Paris 235, 134–136 (1952).

The author finds an approximation for small t to the inverse Laplace transform of $K_0(r\sqrt{p})[pK_0(\sqrt{p})]^{-1}$, $r > 1$, by the standard method of using the asymptotic form of K_0 for large variable. He also estimates the error. *A. Erdélyi.*

Rizza, Giovanni Battista. Sull'estensione al caso di n variabili del metodo diretto degli operatori funzionali. Atti Accad. Ligure 7, 181–202 (1951).

Sbrana and his school [these Rev. 10, 701, 702; 12, 95, 416; 13, 133] applied operational calculus for the solution of boundary value problems. The operational calculus is based on a suitable integral representation of functions (Fourier integral and the like) and an "integral representation of zero" (which is useful in connection with the boundary conditions). In the present paper the corresponding developments are carried out in n variables, and the resulting operational calculus is used for the integration of Laplace's and Poisson's equation in $n(\geqq 4)$ variables.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Berman, D. L. On the estimation of the derivatives of an algebraic polynomial. Doklady Akad. Nauk SSSR. (N.S.) **84**, 197–200 (1952). (Russian)

The author deduces from results of a preceding paper [same Doklady **73**, 249–252 (1950); these Rev. **12**, 111] that a polynomial $P(x)$ of degree n or less, bounded by 1 at the points $\cos k\pi/n$ ($k=0, 1, \cdots, n$), satisfies $|P'(x)|\leqq cn^2$ on $(-1, 1)$, with a universal constant c. [Actually $c=1$: see Duffin and Schaeffer, Trans. Amer. Math. Soc. **50**, 517–528 (1941); these Rev. **3**, 235.] The main part of the paper is devoted to generalizations of this result to k-dimensional polynomials and partial derivatives. The full statements are too long to reproduce here. *R. P. Boas, Jr.*

Mohr, Ernst. Der sogenannte Fundamentalsatz der Algebra als Satz der reellen Analysis. Math. Nachr. **6**, 65–69 (1951).

By elementary methods it is shown that every polynomial with real coefficients may be expressed as the product of linear and quadratic factors. The proof is by real variables, and does not make use of trigonometric or logarithmic functions. *A. C. Schaeffer* (Madison, Wis.).

Mohr, Ernst. Nachtrag zu meiner Arbeit: Beweis des sogenannten Fundamentalsatzes der Algebra im reellen Gebiete. J. Reine Angew. Math. **189**, 250–252 (1952).

A statement made in the earlier paper referred to in the title [same J. **184**, 175–177 (1942); these Rev. **5**, 169] is discussed in detail. *A. C. Schaeffer* (Madison, Wis.).

Stoner, Wm. J. Theorem on the zeros of polynomials. Proc. Iowa Acad. Sci. **58**, 311–312 (1951).

By use of Rouché's theorem, the author proves that the polynomial $P(z)=a_0+a_1z+\cdots+a_nz^n$, $a_n\neq 0$, has all its zeros in or on the circle $|z|=K=(|a_0|+|a_1|+\cdots+|a_{n-1}|)/|a_n|$ when $1\leqq K$, and in or on the circle $|z|=K^{1/n}$ when $K<1$. *M. Marden* (Milwaukee, Wis.).

Kuipers, L. Note on the location of zeros of polynomials. II. Simon Stevin **28**, 193–198 (1951).

This is a continuation of the author's previous paper [Nederl. Akad. Wetensch., Proc. **53**, 482–486 (1951); these Rev. **12**, 175]. With $f(z)=(z-a_1)(z-a_2)\cdots(z-a_n)$ and $g(z)=(z-b)^n$, the author shows that no zero of $f(z)+kg(z+p)$, where $|k|\leqq 1$ and $0\leqq p<\min|b-a_j|$ for $j=1, 2, \cdots, n$, can lie in the region common to all the hyperbolas $|z-a_j|-|z-b|>p$. Likewise, no point common to all the circles

$$|z-b|-p_j|z-a_j|<0$$

where $0<p_j\leqq 1$ for $j=1, 2, \cdots, n$, can be a zero of the polynomial $p_1p_2\cdots p_nf(z)+kg(z)$. As pointed out by the author, these two results are generalizations of those by G. de Sz. Nagy [Bull. Amer. Math. Soc. **53**, 1164–1169 (1947); these Rev. **9**, 237]. Additional results are given for certain linear combinations of $f(z)$ and its first and second derivatives $f'(z)$ and $f''(z)$. For example, if all the zeros of $f(z)$ lie in the circle $|z-c|\leqq d<1$, then all the zeros of $f'(z)+nf(z)$ lie in the circle $|z-c|\leqq d+1$. All the theorems are proved by use of elementary inequalities. Reviewer's note: Sharper results than the last theorem quoted are obtainable by use of Grace's Theorem [see M. Marden, Geometry of the zeros . . . , Amer. Math. Soc., New York, 1949, pp. 62–63; these Rev. **11**, 101]. *M. Marden.*

Bonsall, F. F., and Marden, Morris. Zeros of self-inversive polynomials. Proc. Amer. Math. Soc. **3**, 471–475 (1952).

Nouvelles démonstrations, fondées sur les propriétés des fonctions analytiques, d'un théorème dû à Cohn [Math. Z. **14**, 110–148 (1922)]: Si le polynome

$$g(z)=b_0+b_1z+\cdots+b_mz^m$$

est réciproque, c'est à dire si $b_{m-k}=u\bar{b}_k$ ($k=0, 1, 2, \cdots, m$) avec $|u|=1$, il a, dans $|z|<1$, le même nombre de zéros que $G(z)=m\bar{b}_m+(m-1)\bar{b}_{m-1}z+\cdots+\bar{b}_1z^{m-1}$. Corollaires. *J. Lelong* (Lille).

Skovgaard, Helge. On the greatest and the least zero of Laguerre polynomials. Mat. Tidsskr. B. **1951**, 59–66 (1951).

Using elementary estimates of the power sums, the author obtains inequalities for the least zero x_1 and for the largest zero x_n of $L_n(x)$ and also for the last maximum ξ of $e^{-x}[L_n(x)]^2$, $\xi>x_n$. We mention the following results:

$$1.4453(n+\tfrac{1}{2})^{-1}<x_1<1.455\left[n+\tfrac{1}{2}-\frac{3}{44}(n+\tfrac{1}{2})^{-1}\right]^{-1},$$

$$4n-\alpha(4n)^{1/3}-6<x_n<4n-\beta(4n)^{1/3}+4$$

where

$$\alpha=2\cdot 3^{1/3}+3^{1/2},\quad \beta=2+3^{1/3}.$$

G. Szegö (Stanford University, Calif.).

Tietz, Horst. Eine Rekursionsformel der Faberschen Polynome. J. Reine Angew. Math. **189**, 192 (1951).

Let $P(t)=a_0+a_1t+a_2t^2+\cdots$ be a given power series; the associated Faber polynomials $P_n(x)$ are defined by the coefficient of t^{n-1} in the expansion of

$$t^{-1}\frac{d}{dt}\log\left[1+tP(t)-tx\right].$$

The following recursion formula holds:

$$P_{n+1}(x)+(a_0-x)P_n(x)+a_1P_{n-1}(x)+\cdots$$
$$+a_{n-1}P_1(x)+(n+1)a_n=0$$

for which the author gives a simple proof based on the use of power sums. *G. Szegö* (Stanford University, Calif.).

Mazzoni, Pacifico. Sulle proprietà di alcuni polinomi. Ricerca, Napoli **2**, no. 1, 22–27; no. 2, 18–24; no. 3–4, 14–21 (1951).

The function $x^{n+\frac{1}{2}}J_{n+\frac{1}{2}}(x)$ can be written in the form $U_n\sin x+V_n\cos x$ where U_n and V_n are certain polynomials in x. Starting out from the classical integral representation of J the author derives in a direct manner various properties of U_n and V_n, like differential equations, recurrence formulas, continued fraction expansions, etc. *G. Szegö.*

Special Functions

✓ ★**Lösch, Friedrich, und Schoblik, Fritz. Die Fakultät (Gammafunktion) und verwandte Funktionen mit besonderer Berücksichtigung ihrer Anwendungen.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1951. vi+205 pp. \$4.03.

This book presents the gamma function, and some related functions, in a manner suitable for providing sufficiently specific and detailed information for applied mathematicians, physicists, and engineers. In practical work it is

designed to be used in conjunction with the Tables by Jahnke and Emde. The book derives added importance from its being the only recent and comprehensive monograph on its subject.

The late Friedrich Schoblik (the first name as given in the book is in error) was commissioned to write this book, and the complete manuscript prepared by him was in the hands of the publishers some years ago. Since Schoblik was not available, F. Lösch was asked to undertake the final revision; and since he found some changes in the organisation and presentation of the material desirable, he decided to rewrite the book. On the whole (Lösch explains in the preface) he retained the material provided by Schoblik.

There are three chapters. The first, on the gamma function, occupies about one half of the book, and the other two, on incomplete gamma functions and on applications, respectively, about one quarter each. The most conspicuous feature of the presentation is the tendency to avoid integral representations. The gamma function is defined by its product representations, and Euler's integral occurs for the first time on p. 62 (of the 100 pp. devoted to the gamma function). This makes for a presentation which is methodical and sufficiently different from the usual presentation to please the advanced reader. It does not seem to be the most natural presentation, though, and perhaps it will be somewhat confusing for the beginner. Stirling's series, for instance, is derived from the difference equation satisfied by $\log (z!)$ by means of certain theorems on difference equations (proved in the book), while the derivation from Binet's integral by means of Watson's lemma seems simpler and more natural. The book follows Jahnke-Emde and other authors in denoting $\Pi(z)=\Gamma(z+1)$ by $z!$.

A brief description of the contents follows. Chapter I. The factorial function. §1. Product representations. $\Psi(z)$ and its derivatives. Connection between the factorial function and circular functions. Special values. The Pincherle-Mellin formula for the asymptotic behaviour of the factorial function in a vertical strip. Descriptive properties for real and complex z (including relief diagrams of $z!$ and of $(z!)^{-1}$. Multiplication formulas. §2. Lemmas on difference equations. Asymptotic estimates, Stirling's formula. Some auxiliary functions. Stirling's series and numerical applications. §3. Power series. Factorial series. The expansions of Gudermann, Binet, and Burnside. Fourier expansions. §4. Euler's integral of the second kind. Hankel's contour integral. Integrals for $\Psi(z)$, $\log (z!)$, and for some auxiliary functions. §5. Evaluation of certain definite integrals in terms of the factorial function. The beta function. §6. Linear difference equations of the first order with rational coefficients.

Chapter II. The incomplete factorial functions. The notation used in this book is

$$(z, \rho)! = \int_0^\rho e^{-t} t^z dt, \quad Q(z, \rho) = \int_\rho^\infty e^{-t} t^z dt.$$

§1. Definition and basic properties. Series expansions. Asymptotic properties for large z. §2. Incomplete factorials regarded as functions of ρ. Series. Asymptotic properties. Definite integrals. §3. The exponential integral. The sine and cosine integrals. Integrals and Fourier expansions. §4. The error function. Fresnel integrals. Definite integrals. §5. Hermite polynomials. Recurrence relations, difference equations. Generating functions. Orthogonal properties, expansion theorem. Integral representations. Parabolic cylinder functions and Whittaker's confluent hypergeometric functions.

Chapter III. Applications. The applications presented in about 50 pages show a pleasing diversity and are well designed to introduce the reader to many different uses of factorial functions. The fields touched upon include statistics, probability theory, conduction of heat, electrostatics and electromagnetic theory, actuarial mathematics, optics, and civil engineering.

The book is well written, in a straightforward and lucid style. Apart from standard theorems (for which reference is made to standard textbooks on advanced calculus and complex variable theory) almost everything is proved in the book. A clear plan, a detailed table of contents, and a sufficient index make the book suitable to be used as a handbook. There are some points open to criticism, though. An expansion theorem for Hermite polynomials (which does not really belong to a book of this nature) is called "the" expansion theorem. It is neither the oldest, nor the newest, nor the most general expansion theorem. Certain topics seem to be excluded simply because they do not fit into the plan. For instance, the incomplete factorials are always considered either as functions of z, or as functions of ρ, and hence the asymptotic formula for the case when both z and ρ are large does not fit into the plan and is accordingly omitted. Throughout the book references to textbooks on various branches of analysis, and also to memoirs and papers used in the text are given, but there is nowhere a bibliography, or even a list of monographs on the gamma function, and the reader has no means of tracking down information which could not be included in the present book. Lastly, parabolic cylinder functions and confluent hypergeometric functions, and to some extent even Hermite polynomials, are not indispensable in a book of this kind. On the other hand, a brief section of Riemann's and Hurwitz's zeta functions, and their generalisation by Lerch, should have been included because they occur in several problems in physics and chemistry, where their connection with the functions related to the gamma function is not recognized.

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Sulla norma del complemento $\Gamma(\alpha, x)$ della funzione gamma incompleta per $\alpha=-\frac{1}{2}$. Boll. Un. Mat. Ital. (3) 6, 27–29 (1951).

Tricomi has shown that

$$N(\alpha, x) = \Gamma(\alpha, ix)\cdot\Gamma(\alpha, -ix)$$
$$= \frac{1}{\Gamma(2-2\alpha)} L_x\left[\frac{t^{-2\alpha}}{t+i} F\left(1, 1-\alpha; 2-2\alpha; \frac{t}{t+i}\right) + \frac{t^{-2\alpha}}{t-i} F\left(1, 1-\alpha, 2-2\alpha, \frac{t}{t-i}\right)\right],$$
$$\Re\alpha<1, \quad \Re x>0, \quad L_x[F(t)] = \int_0^\infty e^{-xt} F(t) dt$$

[same Boll. 4, 341–344 (1949); these Rev. 11, 593].

For $\alpha=-\frac{1}{2}$ the author derives

$$N(-\tfrac{1}{2}, x) = 2^{3/2} x^{-1} L_x[t(1+t^2)^{-1/2}[(1+t^2)^{1/2}+1]^{-3/2}],$$
$$N(-\tfrac{1}{2}, x) = 4(2\pi)^{1/2} x^{-1} L_{2\sqrt{x}}[J_{3/2}(t^2)].$$

S. C. van Veen (Delft).

Tricomi, Francesco G. Sulla funzione gamma incompleta. Ann. Mat. Pura Appl. (4) 31, 263–279 (1950).

In this paper the author proposes to give a systematic treatment of the theory and applications of the important

Gamma function, defined by

$$\gamma(\alpha, x)=\int^x e^{-t}t^{\alpha-1}dt$$

and the related fundamental function $\gamma^*(\alpha, x)$, defined by

$$\gamma^*(\alpha, x)=\gamma(\alpha, x)/x^\alpha\Gamma(\alpha)$$

which is an integral function of α and x. Among many well known results we note some new results obtained by the author by means of expressions in 1) Bessel functions: e.g.

$$\gamma(\alpha, x)=x^{\alpha/2}\int_0^\infty e^{-t}t^{\alpha/2-1}J_\alpha(2\sqrt{xt})dt, \quad (\Re\alpha>0, \Re x>0)$$

$$\gamma^*(\alpha, -x)=e^x x^{-\alpha/2}\sum_{n=0}^\infty(-1)^n e_n(-1)x^n J_{n+\alpha}(2\sqrt{x}),$$

where $e_n(-1)=\sum_{h=0}^\infty(-1)^h/h!$.

By putting $\alpha=\frac{1}{2}$, $x=u^2$ we find

$$\text{Erfi}\,(u)=\int_0^u e^{t^2}dt=\tfrac{1}{2}\pi^{1/2}u\gamma^*(\tfrac{1}{2}, -u^2)$$

$$=\tfrac{1}{2}\sqrt{\pi u}\,e^{u^2}\sum_{n=0}^\infty(-1)^n e_n(-1)u^n J_{n+\frac{1}{2}}(2u).$$

The last expansion is very well adapted for the numerical computation of Erfi (u), a function difficult of access for the rest. For $x=1$, 8 terms will suffice for computing the value in six places. Related results are:

$$\text{Erf}\,(x)=\int_0^x e^{-t^2}dt=(\tfrac{1}{2}\pi)^{1/2}\sum_{n=1}^\infty(-1)^{[n/2]}I_{n-\frac{1}{2}}(x^2),$$

$$\text{Erfi}\,(x)=(\tfrac{1}{2}\pi)^{1/2}\sum_{n=0}^\infty(-1)^{[n/2]}I_{n+\frac{1}{2}}(x^2).$$

2) Laguerre functions: e.g.

$$\Gamma(\alpha, x)=\Gamma(\alpha)-\gamma(\alpha, x)$$

$$=e^{-x}x^\alpha\sum_{n=0}^\infty\frac{L_n^{(\alpha)}(x)}{n+1}=\frac{e^{-x}}{\Gamma(1-\alpha)}\sum_{n=0}^\infty\frac{L_n^{(-\alpha)}(x)}{n+1-\alpha}.$$

In the end we note the remarkable relations

$$\gamma^*(\alpha, x)=\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 e^{-tx}t^{\alpha-\beta-1}\gamma^*(\beta, x-tx)dt$$

$$(\Re\alpha>\Re\beta>-1, \alpha\beta\neq0)$$

and the addition theorem:

$$\gamma(\alpha, x+y)=e^{-y}\sum_{n=0}^\infty\frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)}\gamma(\alpha-n, x)\frac{(-y)^n}{n!}.$$

S. C. van Veen (Delft).

Tricomi, Francesco. Sviluppo in serie asintotica del rapporto $\Gamma(z+\alpha):\Gamma(z+\beta)$. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 343–351 (1950).

The author derives the asymptotic expansion

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim\sum_{n=0}^\infty C_n(\alpha-\beta, \beta)z^{\alpha-\beta-n}$$

where C_n may be defined by the recurrence relation

$$C_0=1,$$

$$C_n(\alpha', \beta)=\frac{1}{n}\sum_{m=0}^{n-1}\left\{\binom{\alpha-m}{n-m}-(-1)^{n+m}\alpha'\beta^{n-m}\right\}C_m(\alpha', \beta) \quad (n=1, 2, \cdots).$$

This expansion is valid throughout the z-plane, cut along an arbitrary curve from $z=0$ to $z=+\infty$, with the exception of the values $z=-\alpha, -\alpha-1, \cdots$ and $z=-\beta, -\beta-1, \cdots$. The author notes the special case $\alpha=\frac{1}{2}$, $\beta=1$, $z=n$, from which he derives the approximate expression for π:

$$\pi=\frac{1}{n}\left\{\frac{2^n n!}{1\cdot3\cdots(2n-1)}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+e_n\right)\right\}^2, \quad e_n=O(n^{-3}).$$

[Reviewer's remark: An expansion of the same character, but of more regular form

$$\pi=\frac{2}{2n+1}\left\{\frac{2^n n!}{1\cdot3\cdots(2n-1)}\right\}^2$$

$$\times\left(1+\frac{1^2}{2\cdot(2n+3)}+\frac{1^2\cdot3^2}{2\cdot4\cdot(2n+3)(2n+5)}+\cdots\right)$$

may be derived from a classical result of Binet for $\Gamma(2z)/\{\Gamma(z)\}^2$ [cf. Whittaker and Watson, Modern analysis, 4th ed., Cambridge, 1927, ch. XII, p. 263, Misc. ex. 42].]

S. C. van Veen (Delft).

Ou, Vincent Tchen-yang. Sur les fonctions hypergéométriques de plusieurs variables. C. R. Acad. Sci. Paris 234, 1524–1526 (1952).

The integral $c\int u^\alpha(u-1)^\beta(u-x)^\gamma(u-y)^\delta du$ taken over all contours closed on the Riemann surface of the integrand, determines a three (complex) dimensional linear space of analytic functions of x and y. Alternatively, the author regards three linearly independent branches of the integral as homogeneous coordinates in a two (complex) dimensional projective space. $\alpha, \beta, \gamma, \delta$ are assumed real. As x and y vary through real values, the (complex) point in the projective 2-space describes a number of homographic transforms of the real (projective) subspace. These are called planar chains and are used to discuss the monodromic group of the system of partial differential equations associated with the integral. There is also a Hermitian quadratic form associated with a quadric which is orthogonal to four chains. If $\alpha, \beta, \gamma, \delta$, $\alpha+\beta+\gamma+\delta$ are non-integer and α' denotes the positive fractional part of α, etc., then the Hermitian form is elliptic if $\alpha'+\beta'+\gamma'+\delta'<1$ or >3 and is hyperbolic if $1<\alpha'+\beta'+\gamma'+\delta'<3$.

A. Erdélyi (Pasadena, Calif.).

Niblett, J. D. Some hypergeometric identities. Pacific J. Math. 2, 219–225 (1952).

The author extends a hypergeometric identity given by Chaundy [J. London Math. Soc. 26, 42–44 (1951), equation (4); these Rev. 12, 410] to generalized hypergeometric functions and discusses a large number of special cases. These lead to interesting transformation and reduction formulas for generalized hypergeometric functions.

A. Erdélyi (Pasadena, Calif.).

Ossicini, Alessandro. Formula e serie di approssimazione asintotica delle funzioni ultrasferiche di seconda specie. Boll. Un. Mat. Ital. (3) 7, 48–53 (1952).

The ultraspherical function of the second kind $Q_n^{(\lambda)}(\cos\theta)$ (θ real) is represented in terms of hypergeometric functions. A resulting integral representation is used in order to derive an asymptotic expansion of $Q_n^{(\lambda)}$ with an explicit estimate of the error terms. The expansion is a counterpart to a classical expansion of Stieltjes.

G. Szegö.

Sharma, A. On certain relations between ultraspherical polynomials and Bessel functions. Bull. Calcutta Math. Soc. 43, 61–66 (1951).

Various relations between Legendre polynomials and Bessel functions have been discussed by S. C. Mitra, B. N. Bose and S. K. Bose. In the present contribution these relations are generalized by replacing the Legendre polynomials by ultraspherical ones. Also certain relations due to Gegenbauer, involving Bessel functions only, are proved in a new manner. *G. Szegö* (Stanford University, Calif.).

Palamà, Giuseppe. Contributo alla ricerca di relazioni fra classici polinomi. Rivista Mat. Univ. Parma 2, 383–402 (1951).

Relations are discussed connecting some classical polynomials (Legendre, Gegenbauer, Jacobi, Laguerre and Hermite polynomials) with each other as well as with hypergeometric and confluent hypergeometric series. Tchebychev polynomials are discussed in detail. *G. Szegö.*

Mukherjee, B. N., and Nanjundiah, T. S. Tschebyscheff polynomials $T_n(z)$ and $U_n(z)$ and functions of the second kind. Proc. Indian Acad. Sci., Sect. A. 35, 19–23 (1952).

Tchebychev polynomials of both kinds and related functions are discussed in particular in their relation to hypergeometric functions. *G. Szegö.*

Bagchi, Hari Das, and Mukherjee, Bhola Nath. A note on the generalised Laguerre polynomial. Proc. Indian Acad. Sci., Sect. A. 35, 53–55 (1952).

Let $f_n(z)$ be an analytic solution of the difference equation $(n+1)f_{n+1}(z)-(2n+\alpha+1-z)f_n(z)+(n+\alpha)f_{n-1}(z)=0$. A representation of the generating function $\sum h^n f_n(z)$ in terms of the initial functions $f_0(z)$ and $f_1(z)$ is given; $|h|<1$. For a suitable choice of the functions f_0 and f_1 the generating function of the Laguerre polynomials follows. *G. Szegö.*

Gatteschi, Luigi. On the zeros of certain functions with application to Bessel functions. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 224–229 (1952).

Using the classical asymptotic formula of the Bessel functions (with two terms) the author proves the following estimates for the zeros j_{nm} of $J_n(x)$: Let

$$n\geqq 0,\quad j_{nm}>\pi^{-1}(2n+1)(2n+3)$$

and let us denote by x_{nm} the quantity $(m+\frac{1}{2}n-\frac{1}{4})\pi$. Then

$$\left|j_{nm}+\frac{(n,1)}{2x_{nm}}-x_{nm}\right| \leqq \frac{A|(n,1)^2|+B|(n,1)(n,2)|+C|(n,3)|+2(n,1)^2}{\{(2m+n-1)\pi\}^3}.$$

Here $(n,p)=\Gamma(n+p+\frac{1}{2})/p!\Gamma(n-p+\frac{1}{2})$ and

$$A=0.65,\quad B=1.29,\quad C=3.35\quad \text{for}\quad n\leqq 7/2,$$
$$A=0.573,\quad B=1.146,\quad C=11.9\quad \text{for}\quad n>7/2.$$

G. Szegö (Stanford University, Calif.).

Luke, Yudell L. An associated Bessel function. J. Math. Physics 31, 131–138 (1952).

The author defines suitable particular solutions of the differential equation

$$\left(z^2\frac{d^2}{dz^2}+z\frac{d}{dz}-\nu^2-z^2\right)y=z^{\mu+1}e^{-z}.$$

He gives expansions in ascending powers of z, the connection with modified Bessel functions, special formulas when $\mu\pm\nu$ is a negative integer, and when $\mu+1=\nu$ is half of an odd integer. Integrals of the form $\int x^\mu e^{-x}I_\nu(x)dx$ may be expressed in terms of the functions introduced in this paper. The integral $\int_0^\infty e^{-z\cosh t}\sinh(\nu t)dt$ and some related integrals are also discussed. *A. Erdélyi* (Pasadena, Calif.).

Breit, G., and Hull, M. H., Jr. Asymptotic expansion of the irregular Coulomb function. Physical Rev. (2) 80, 561–563 (1950).

Yost, Wheeler and Breit [same Rev. 49, 174–189 (1936)] have shown that the regular Coulomb function F_L may be expanded as an asymptotic power series in the energy, with coefficients which are functions of the distance r and are expressed in terms of $I_n(x)$, the Bessel function of the first kind. In this paper it is shown that the irregular Coulomb function G_L can be expanded as an analogous asymptotic power series, with coefficients which are expressible in terms of $K_n(x)$, the modified Bessel function of the second kind. The form of the coefficients is the same as obtained by Yost, Wheeler and Breit, with the modified Bessel functions of the second kind replacing those of the first kind of the same order. *S. C. van Veen* (Delft).

Abramowitz, Milton. Coulomb wave functions expressed in terms of Bessel-Clifford and Bessel functions. J. Math. Physics 29, 303–308 (1951).

The regular solution of the Coulomb wave equation

$$\frac{d^2y}{d\rho^2}+\left[1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right]y=0 \tag{1}$$

is to be tabulated for $L=0(1)5$, 10, 11, 20, 21; $\rho=0(.2)5$; $\eta=-5(1)5$. The author gives expressions for the regular solution which will be useful outside the range of the tables. The regular solution $y=F_1(\eta,\rho)$ of (1) is normalised in such a manner that as $\rho\to\infty$

$$F_L(\eta,\rho)\sim\sin\left\{\rho-\eta\ln 2\rho-\frac{L\pi}{2}+\arg\Gamma(L+1+i\eta)\right\}.$$

The power series expansion of $F_L(\eta,\rho)$ is

$$F_L(\eta,\rho)=C_L(\eta)\rho^{L+1}\phi_L(\eta,\rho)$$

where

$$C_L=\left\{\frac{2\pi\eta}{\exp(2\pi\eta)-1}\right\}^{\frac{1}{2}}\frac{2^L}{\Gamma(2L+2)}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(1+i\eta)}\right|.$$

The function $\phi_L(\eta,\rho)=u$ satisfies the differential equation

$$\rho u''+(2L+2)u'+(\rho-2\eta)u=0\quad\text{with } u=1 \text{ for } \rho=0.$$

In the first an expansion of u is given in terms of the modified Bessel-Clifford function $E_n(t)=t^{-n/2}I_n(2\sqrt{t})$ when η is large.

$$\phi_L(\eta,\rho)=k(\eta)\left[E_{2L+1}(2\eta\rho)+\sum_{s=1}^{\infty}a_s(\eta)E_{2L-s}(2\eta\rho)\right]$$

where $k^{-1}(\eta)=1/(2L+1)!+\sum_{s=1}^{L}a_s(\eta)/(2L-s)!$ and the coefficients $a_s(\eta)$ are given by recurrence relations. In the second place $\phi_L(\eta,\rho)$ is expanded in terms of the Bessel functions, which expansion is useful when η is small and ρ is large:

$$\phi_L(\eta,\rho)=\sum_{n=0}^{\infty}b_n(\eta)J_n(\rho),\quad b_0=1.$$

The coefficients $b_n(\eta)$ are given by recurrence relations. The last expansion may also be obtained with the aid of the Laplace transform. *S. C. van Veen* (Delft).

Campbell, Robert. Nouvelles équations intégrales pour les fonctions de Lamé. C. R. Acad. Sci. Paris 234, 2515–2517 (1952).

For periodic Lamé functions, $E^N_{m-\frac{1}{2}}(u)$, the author proves the integral equation

$$E^N_{m-\frac{1}{2}}(u)=\lambda_N\int_0^{4K} M(u,u')E^N_{m-\frac{1}{2}}(u')du'$$

where M is either $\rho^{m+\frac{1}{2}}$ or $e^{-qz}\rho^{\frac{1}{2}}J_m(q\rho)$, q is an arbitrary constant and z, ρ are cylindrical coordinates expressed in terms of the coordinates u, u' of cyclides of revolution.

A. Erdélyi (Pasadena, Calif.).

Campbell, Robert. Equations intégrales et fonctions de Lamé. C. R. Acad. Sci. Paris 235, 8–10 (1952).

Further kernels M for the integral equation.

A. Erdélyi (Pasadena, Calif.).

Stoppelli, Francesco. Sulle famiglie di funzioni definite da equazioni differenziali lineari omogenee a coefficienti costanti. Giorn. Mat. Battaglini (4) 3(79), 225–229 (1950).

The author characterizes the class E_n of functions defined almost everywhere on the interval $(0, \lambda)$ and coinciding with a solution of a linear homogeneous differential equation with constant coefficients and of order $\leqq n$ there. He proves the following result: If $\{\psi_\nu(x)\}$ is a sequence of elements of E_n and if there is a measurable function $\psi(x)$ such that

$$\lim_{\nu\to\infty}\int_0^x (x-t)^k\psi_\nu(t)dt=\int_0^x (x-t)^k\psi(t)dt,$$
$$k=0,1,\cdots,n,\quad 0\leqq x\leqq\lambda,$$

then $\psi(x)$ belongs to E_n. *A. Erdélyi* (Pasadena, Calif.).

Benedetti, Carlo. La funzione θ_n collegata alla costante di Eulero-Mascheroni. Period. Mat. (4) 28, 169–174 (1950).

Der Verfasser betrachtet die Folge

$$\theta_n=n\left\{C-\sum_{k=1}^{n-1}\frac{1}{k}+\log n\right\},\quad n=1,2,\cdots,$$

wo C die Euler-Mascheronische Konstante bedeutet. Er zeigt nach zwei Methoden die bekannten Ergebnisse: 1) die Folge θ_n fällt monoton; 2) $\lim_{n\to\infty}\theta_n=\frac{1}{2}$. Weiter betrachtet er die Reihe

$$E_h=\sum_{n=1}^{\infty}\left(\frac{1}{n+h}-\log\frac{n+h+1}{n+h}\right),\quad 0\leqq h\leqq 1.$$

Es wird gezeigt: $E_h<C$; $E_1=C-1+\log 2$. Schliesslich wird eine Tabelle für θ_n, $n=1(1)10$, $20(10)100$ gegeben.

S. C. van Veen (Delft).

Harmonic Functions, Potential Theory

Beckenbach, E. F. On subharmonic, harmonic and linear functions of two variables. Univ. Nac. Tucumán. Revista A. 8, 7–13 (1951).

Extension aux fonctions de deux variables de résultats de T. Radó [Trans. Amer. Math. Soc. 37, 226–285 (1935)] et C. L. Woods [Bull. Amer. Math. Soc. 52, 117–128 (1946); ces Rev. 7, 246]. Parmi les fonctions $f>0$ dans un domaine plan l'auteur caractérise celles qui sont sousharmoniques, harmoniques, ou linéaires, à l'aide de relations simples entre $A_{r,\alpha}(f;x,y)$ et $C_{r,\beta}(f;x,y)$ pour des valeurs convenables de α et β; $(A_{r,\alpha})^\alpha$ (resp. $(C_{r,\alpha})^\alpha$) est la valeur moyenne de f^α sur le cercle (resp. la circonférence) de centre x, y et de rayon r; la définition s'étend au cas $\alpha=0$. Par exemple les f linéaires sont caractérisées par $A_{r,-1}=C_{r,0}$. *J. Deny.*

Kim, E. I. On a general boundary problem of a harmonic function. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 147–158 (1952). (Russian)

Let S_0 be a connected region, bounded by "smooth", closed, nonintersecting contours $L_0,\cdots,L_p$, L_0 containing the other contours; let S_p be the region interior to L_p, $S=S_1+\cdots+S_p$, $\sigma=\bar{S}$, $\sigma^-=$complement of $\sigma+L_0$; the origin of coordinates (x,y) is taken in S_0. The author solves the following problem. To find a function $u(x,y)$, harmonic in S, continuous in σ, such that

$$\left(\frac{\partial u}{\partial n}\right)_i=H_q\left(\frac{\partial u}{\partial n}\right)_e+f_q(s)\quad\text{on}\quad L_q\ (q=1,\cdots,p),$$
$$\sum_{k=0}^{m}\sum_{j=0}^{k}a_{kj}(s)\left(\frac{\partial^k}{\partial x^{k-j}\partial y^j}\right)_i=F(s)\quad\text{on}\quad L_0$$

where the H_q are assigned nonnegative constants, $f_q(s)$, $F(s)$ are continuous functions of arc, the $a_{kj}(s)\epsilon\mathrm{Lip}$, while $|\sum_{j=0}^{k}(i)^j a_{mj}(s)|>0$. The methods used are of the type due to D. I. Sherman [Izvestiya Akad. Nauk SSSR 10, 121–134 (1946); these Rev. 8, 66]. *W. J. Trjitzinsky.*

Soudan, Robert. Indéformabilité d'un corps à potentiel polyharmonique constant. Arch. Sci. Soc. Phys. Hist. Nat. Genève 5, 5–18 (1952).

The author considers an analytic homogeneous body V of density δ and its polyharmonic potential given by $U(P)=\delta\int_V v_n(M,P)d\tau_M$, where $d\tau$ is the element of volume and $V_n(M,P)=\sum_{\alpha=-1}^{2n-2}C_\alpha\overline{MP}^\alpha$. His object is to study the determination of the body by its potential at exterior points, and he shows that if $n>1$ (to rule out the case of the Newtonian potential) and if C_α is different from zero for some odd α, then there can be no analytic deformation of the body leaving the potential unaltered in the exterior of the body. The proof proceeds by showing that if such a deformation exists, then there is a single layer on the surface of V which gives to within a multiplicative constant the same potential as V, and this is shown impossible by an analysis of the Green's function. *J. W. Green.*

Myškis, A. D. A theorem on the convergence of sequences of functions. Uspehi Matem. Nauk (N.S.) 7, no. 1(47), 186–190 (1952). (Russian)

Generalizing a known result concerning harmonic functions, the author shows that if $u_1, u_2,\cdots$ is a sequence of twice continuously differentiable functions of $n\geqq 2$ variables in a domain G, for which $\sup_i\int_G|\Delta u_i|^\alpha dG=M<\infty$, where Δ is the Laplace operator and α is a constant $>n/2$, and if the sequence converges in the mean of order 1, then the sequence converges uniformly on any compact subset $F\subset G$.

E. F. Beckenbach (Los Angeles, Calif.).

Frostman, Otto. Distributions de masses normées par la métrique de L^p. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 21, no. 13, 1–11 (1951).

Let μ_k denote a distribution of mass on a fixed bounded domain D in E_n. Let $U_k^{(\alpha)}(X)$ denote the potential of order

α, $0<\alpha<m$, [Riesz, Acta Math. 81, 1-223 (1949); these Rev. 10, 713], and let

$$\|\mu_k\|_{\alpha,p}=\left[\int_{E_n} U_k^{(\alpha)}(X)^p dX\right]^{1/p}, \quad p\geqq 1.$$

The author notes that for $\alpha=p=2$, $n=3$, and $\int_{E_n} d\mu_k=0$, $k=1, 2, \cdots$, the weak convergence of μ_k to μ implies $\lim_k \|\mu_k-\mu\|_{2,2}=0$. The author extends this result as follows. If $\{\mu_k\}$ is a sequence of distributions of mass on D, if $\int_{E_n} d\mu_k=0$, $k=1, 2, \cdots$, if $\mu_k\to\mu$ weakly, then $\|\mu_k-\mu\|_{\alpha,p}\to 0$ for $n/(n+1-\alpha)<p<n/(n-\alpha)$. If, in addition,

$$p>\max\,[1, n/(n+1-\alpha)],$$

then there exist constants $C_j=C_j(D;\alpha,p)$, such that for α, p fixed, for $p<r$, $\beta<\alpha$,

$$\|\mu\|_{\alpha,p}\leqq C_1\|\mu\|_{\alpha,r}, \quad \|\mu\|_{\alpha,p}\leqq C_2\|\mu\|_{\beta,p}.$$

The author's proofs make extensive use of the formulas of composition of α-potentials and of the Hölder inequality.
M. Reade (Ann Arbor, Mich.).

Nikol'skiĭ, S. M. On the Dirichlet problem. Doklady Akad. Nauk SSSR (N.S.) 83, 23–25 (1952); erratum 84, 652 (1952). (Russian)

As the Hadamard function $f(\vartheta)=\sum_{k=1}^{\infty}(\cos k^4\vartheta)/k^2$ shows, the solution $u(\rho,\vartheta)$ of the Dirichlet problem for the unit circle σ corresponding to a continuous boundary function f may be such as to make the Dirichlet integral $D[u]$ infinite. The author characterizes those boundary functions for which the Dirichlet problem for σ can be solved by variation of the Dirichlet integral, in the following two theorems (stated without proof). Theorem 1: If u is harmonic and summable on σ and $D[u]<\infty$, then, for almost all ϑ, $\lim_{\rho\to 1} u(\rho,\vartheta)=f(\vartheta)$, where $f\epsilon L^2(0, 2\pi)$ and satisfies

$$(*)\qquad \left[\int_0^{2\pi} |f(\vartheta+h)-f(\vartheta)|^2 d\vartheta\right]^{1/2}\leqq M|h|^{1+\epsilon}$$

for $\epsilon=0$. Theorem 2: If $f\epsilon L^2(0, 2\pi)$ and satisfies (*) for $\epsilon>0$, then the function f is continuous and the solution of the Dirichlet problem on σ for the boundary function f has a finite Dirichlet integral. Further properties of functions f satisfying (*) are developed and generalizations made to half-spaces in n dimensions.
M. G. Arsove.

Tolstov, G. P. On bounded functions satisfying Laplace's equation. Mat. Sbornik N.S. 29(71), 559–564 (1951). (Russian)

It is known that if a real-valued function $u(x_1, \cdots, x_n)$ is continuous on a closed set G and $\partial^2u/\partial x_1^2, \cdots, \partial^2u/\partial x_n^2$ exist, their sum adding up to zero at each interior point of G (i.e., the Laplacian $\Delta u=\sum_{i=1}^{n}\partial^2u/\partial x_i^2=0$) then u is analytic at each interior point of G. The author proves further that the same conclusion continues to hold if one merely assumes in the hypothesis, instead of continuity, that u is bounded in G. The proof consists in showing that the seemingly weaker form of the hypothesis implies the stronger form. The author remarks that his method of proof is restricted to $n=2$, and is not directly applicable to $n>2$.
J. B. Diaz (College Park, Md.).

Birindelli, Carlo. Nuova trattazione di problemi al contorno di uno strato, per l'equazione di Poisson in tre variabili. III. Rivista Mat. Univ. Parma 2, 337–364 (1951).

This is the third part of a three-part work on the solution of Poisson's equation, $\Delta u=f(x, y, z)$, in the slab $0<z<a$ by a series expansion method. For a more detailed statement of the problem and description of the expansion method, see the reviews of the first two parts [same Revista 2, 77–102, 235–263 (1951); these Rev. 13, 131, 555]. Part III is devoted to showing that the alleged solution, whose series expansion was determined in Part I and which was shown in Part II to satisfy the given boundary conditions, does indeed satisfy the given differential equation. The proofs depend upon complicated and lengthy analysis of series of trigonometric and Bessel functions, and to attempt to sketch them or the actual existence and uniqueness theorems proved here would be impractical. For example, the final theorem, the principal one of the paper, requires one and one-half pages to state.
J. W. Green.

Pham, Mau Quan. Sur une solution de l'équation d'ondes relative à un espace riemannien simplement harmonique. C. R. Acad. Sci. Paris 234, 2329–2331 (1952).

In a Riemannian space V_n the 'wave equation' is taken to be $\partial^2u/\partial t^2-\Delta_2 u=0$ where Δ_2 is the Laplace operator for V_n. This equation is considered for a V_n which is harmonic, and in particular for an odd-dimensional V_n which is simply harmonic. In the latter case a solution is given which depends upon two arbitrary functions of $s+t$ and $s-t$, where s is the geodesic distance in V_n.
A. G. Walker.

Differential Equations

✶Titt, Edwin W. Linear differential equations, ordinary and partial. Part I. Ordinary equations. Mathematics Research and Publishing Company, Austin, Texas, 1951. vii+222+11 pp. \$3.00.

Contents: I) Ordinary differential equations of first order; II) The ordinary linear differential equation; III) Systems of linear equations with constant coefficients; IV) Bessel functions; Appendix I, II (Table of integrals; Summary of formulae).

Ch. I contains standard material. Ch. II, III contain explicit formulae for solving equations up to fourth order and systems of second order equations. The treatment differs from the conventional one, being based on the use of integrating factors derived from the adjoint equation. This is intended to prepare the reader for similar methods in partial differential equations, to be treated in Part II; it is also claimed that it involves shorter tables than does the use of Laplace transforms. Ch. IV treats Bessel's equation of order 0 and 1; for example, the zero-order Bessel equation is written as $(xy')'=-xy$, and converted to the integral equation $y(x)=y(a)+ay'(a)\log(x/a)+\int_a^x\xi\log(\xi/x)y(\xi)d\xi$. Series expansions and other properties of Bessel functions are derived from integral equations of this kind; this treatment is again chosen to prepare for Part II.

Comment on the paedagogic merits of the approach chosen in Ch. II–IV must await the appearance of Part II; as regards the practical merits of the methods of Ch. II, III, which involve an elaborate array of formulae and tables, the reviewer considers that the more usual methods (Laplace transform or Heaviside calculus) are easier to learn and to use.
G. E. H. Reuter (Manchester).

Anastassiadis, Jean. Sur les solutions entières de quelques équations différentielles. Bull. Sci. Math. (2) 76, 57–64 (1952).

An entire function $f(z)$ of finite order ρ is of type mean σ [type mean σ complete] if $|f(z)|<\exp\{(1+\epsilon)\sigma|z|^\rho\}$ for

$|z|$ sufficiently large, and if $|f(z)|>\exp\{(1-\epsilon)\sigma|z|^\rho\}$ for an infinite set of values of z $[|z|]$ increasing indefinitely in magnitude. Information on solutions of the differential equation

$$(1)\quad \frac{d^n w}{dz^n}+P_1(z)\frac{d^{n-1}w}{dz^{n-1}}+\cdots+P_{n-1}(z)\frac{dw}{dz} = P_n(z)+P_{n+1}(z)w+\cdots+P_{n+k}(z)w^k$$

is obtained. The main results are: (a) If $P_i(z)$ are polynomials, then equation (1) can have no solution of type mean σ if $k>1$; (b) if, for each i, $P_i(z)$ is an entire function of order ρ_i and mean type σ_i and if $\rho_{n+k}>\rho_i$ for $i=1,\cdots,n+k-1$, then equation (1) can admit no solution of type mean σ complete if $k>1$. *F. G. Dressel* (Durham, N. C.).

Sysoev, A. E. Some cases of integrability of differential equations of the 1st order. Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 175–179 (1952). (Russian)

Let $F(x,y)$, $\varphi(x,y)$, $\psi(x,y)$ be functions of x, y which are pairwise independent and let $\bar{F}(\varphi,\psi)$ be the function obtained from $F(x,y)$ by making the substitution $\varphi=\varphi(x,y)$, $\psi=\psi(x,y)$. The author calls $F(x,y)$ homogeneous of degree n with respect to $\varphi(x,y)$, $\psi(x,y)$ if $\bar{F}(t\varphi,t\psi)\equiv t^n\bar{F}(\varphi,\psi)$, and remarks that necessary and sufficient for this to hold is $\partial(F/\varphi^n,\psi/\varphi)/\partial(x,y)=0$. He applies this result to find equations of the form

$$y'=Q(x)y+M(x)y^m+N(x)y^n+\cdots+P(x)y^p$$

which have a general solution of the form $G(\varphi)=cF(\psi/\varphi)$. *M. Golomb* (Lafayette, Ind.).

Latyševa, K. Ya. On the general solution in finite form of linear differential equations. Ukrain. Mat. Žurnal 1, no. 3, 81–100 (1949). (Russian)

The author derives conditions too lengthy to state here under which the linear differential equation

$$\sum_{r=1}^{n}P_r(x)y^{(n-r)}(x)=0$$

with polynomial coefficients has solutions of the form $y=e^{Q(x)}x^\alpha S(x)$, where $Q(x)$ and $S(x)$ are polynomials in x or x^{-1}; also conditions under which the general solution of the equation is a linear combination of such solutions. *M. Golomb* (Lafayette, Ind.).

Malkin, I. G. On the construction of Lyapunov functions for systems of linear equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 239–242 (1952). (Russian)

Let $\dot{x}_s=p_{s1}x_1+\cdots+p_{sn}x_n$ be a system of differential equations, p_{sj} being continuous bounded functions of t in $[0,\infty)$. Let $x^0_{sj}(t,t_0)$ be a fundamental system of solutions, $x^0_{sj}(t_0,t_0)=\delta_{sj}$ and assume $|x^0_{sj}(t,t_0)|<Me^{-\alpha(t-t_0)}$ for $t\geqq t_0\geqq 0$, M, α positive constants. Let $W(t,x_1,\cdots,x_n)$ be any positive definite form of degree m in $x_1,\cdots,x_n$ whose coefficients are continuous bounded functions of t. Then

$$V=\int_t^\infty W(\tau,y_1,\cdots,y_n)d\tau,$$

where $y_s=x^0_{s1}(\tau,t)x_1+\cdots+x^0_{sn}(\tau,t)x_n$, is a positive definite function of Lyapunov, which is a form of degree m in $x_1,\cdots,x_n$ with bounded coefficients, satisfying the equation $dV/dt=-W$. This theorem generalizes previous results by Lyapunov and Malkin. *J. L. Massera* (Montevideo).

Četaev, N. G. On unstable equilibrium in certain cases when the force function is not maximum. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 89–93 (1952). (Russian)

Certain criteria of instability based on the idea of Lyapunov's functions, which were previously proved by the author [N. G. Četaev, Stability of movement, Gostehizdat, Moscow, 1946], are here applied to establish the instability of equilibrium points of dynamical systems in several cases when the force function is not maximum. *J. L. Massera* (Montevideo).

Massera, José L. Remarks on the periodic solutions of differential equations. Bol. Fac. Ingen. Montevideo 4 (Año 14), 37–45 = Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 2, 43–53 (1950). (Spanish)

Systems $\dot{x}=f(x,t)$, $x=(x_1,\cdots,x_n)$, $f=(f_1,\cdots,f_n)$ are considered where the right-hand side of the equation is periodic in t, with period T. The author points out that if $n\geqq 2$, such a system may well admit periodic solutions whose period T' is incommensurable with that of the system. Such a solution is a simple closed curve, tangent to the given field, passing through no singular point of the system (i.e. $f\neq 0$), and along which f does not depend on t. These conditions are essentially sufficient. Applications of this result are made to systems of a more special type. *F. Bohnenblust.*

Staržinskiĭ, V. M. Sufficient conditions for stability of a mechanical system with one degree of freedom. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 369–374 (1952). (Russian)

By considering the behavior of a suitable quadratic form, $V(t)=ax^2+2bxy+cy^2$, where $\dot{x}=y$, $\dot{y}=-q(t)x-p(t)y$, a method first introduced by Liapounoff, the author presents some sufficient conditions for the boundedness of the solutions of $\ddot{x}+p(t)\dot{x}+q(t)x=0$. As an interesting application he considers the case where $p(t)=a$, $q(t)=1+r\cos 2t/k$, $0<r<1$. *R. Bellman* (Santa Monica, Calif.).

Serov, M. I. Remark on the number of zeros of the solution of a linear differential equation of the second order. Uspehi Matem. Nauk (N.S.) 6, no. 6(46), 182–183 (1951). (Russian)

The equation $y''+q(x)y=\lambda y$ is considered with $q(x)$ a complex-valued function of the real variable x and λ complex. Let $u=\operatorname{Im} q(x)$ be continuous in the interval (A,B), finite or infinite, and not equal to a constant in any segment of this interval. Let the sum of the number of points in the interval where u is a maximum or minimum be n. Then it is shown that for no value of λ can a solution of the differential equation have more than $2n+3$ zeros in the interval (A,B). The proof uses the equation

$$\operatorname{Im}\lambda\int_a^b|y_0|^2dx=\int_a^b u(x)|y_0|^2dx,$$

valid for a solution $y_0(x)$ which vanishes at a and b. *N. Levinson* (Cambridge, Mass.).

Filippov, A. F. A sufficient condition for the existence of a stable limit cycle for an equation of the second order. Mat. Sbornik N.S. 30(72), 171–180 (1952). (Russian)

The equation (*) $\ddot{x}+f(x)\dot{x}+g(x)=0$ is considered with $g(x)$ having the same sign as x. The equation $\ddot{x}+\phi(x,\dot{x})\dot{x}+g(x)=0$ is also considered. Sufficient conditions are given for the existence of a stable limit cycle. (The uniqueness of the limit cycle is not considered.) Sufficient conditions for the non-existence of periodic solutions are also given. For $x>0$

let $\int_0^x g(\xi)d\xi = z_1(x)$, and for $x<0$, $\int_0^x g(\xi)d\xi = z_2(x)$. The variable $y=\dot{x}-F(x)$, where $F(x)=\int_0^x f(\xi)d\xi$, is introduced and x is replaced by z_1 for $z_1>0$ and for $x>0$; then (*) becomes $dz_1/dy=F_1(z_1)-y$ where $F_i(z_i)=F(x(z_i))$, $i=1, 2$, and similarly with z_2 for $x<0$. It is these last equations that are used in the proofs. One theorem assumes that, for small z ($z<\delta$), $F_1(z)\leqq F_2(z)$ but not with $F_1(z)=F_2(z)$ for all z, and $F_1(z)<a\sqrt{z}$, $F_2(z)>-a\sqrt{z}$ where $a<\sqrt{8}$. Moreover, if there exists a number z_0 such that $\int_0^{z_0}[F_1(z)-F_2(z)]dz>0$ and, for $z>z_0$, $F_1(z)\geqq F_2(z)$, $F_1(z)>-a\sqrt{z}$, $F_2(z)<a\sqrt{z}$ (where $a<\sqrt{8}$) then (*) has a stable limit cycle.

N. Levinson (Cambridge, Mass.).

Wax, Nelson. On amplitude bounds for certain relaxation oscillations. J. Appl. Physics 22, 278–281 (1951).

The differential equation $\ddot{x}+f(x)\dot{x}+g(x)=0$ has exactly one periodic solution if $f(x)$ and $g(x)$ are suitably restricted to behave qualitatively as $f(x)=x^2-1$, $g(x)=x$. Upper and lower estimates for the amplitude of the periodic solution are obtained in the present paper, in terms of the functions $F=\int f dx$ and $G=\int g dx$. In the particular case of van der Pol's equation and for $\mu=4$, these estimates fix the amplitude between 1.85 and 2.14.

F. Bohnenblust.

Friedlander, F. G. On the asymptotic behaviour of the solutions of a class of non-linear differential equations. Proc. Cambridge Philos. Soc. 46, 406–418 (1950).

This paper discusses asymptotic properties of the solutions of the differential equation $\ddot{x}+x=kf(x, \dot{x}, t)$. As happens in a number of important cases the mean energy balance $k\int f(x, \dot{x}, t)dx/(t_2-t_1)$ depends sensibly only on the amplitude $(x^2+\dot{x}^2)$ for large t_2-t_1 and small k. Whenever this occurs, this fact can be used to obtain approximate results concerning the asymptotic range of variation of the amplitude as t tends to infinity. The object of this paper is to put this argument into precise analytical form.

The function f is assumed to be such that the limit of the time average $AT^{-1}\int f(x, y, t)$, taken from $t=t_1$ to $t=t_1+T$ along $x=A\sin(t+\alpha)$, $y=A\cos(t+\alpha)$, exists as T tends to infinity, depends on A alone and exists uniformly in α, $t_1\geqq 0$ for $0<H\leqq A\leqq K$. This limit is denoted by $\mu(A)$. In addition f is assumed continuous, to satisfy of course a Lipschitz condition, and to be bounded for all t in the ring $H\leqq R\leqq K$, where $R^2=x^2+y^2$. If $x(t)$ is a solution of the differential equation and $A^2(t)$ is equal to $x^2+\dot{x}^2$ then estimates are obtained for the limit superior and the limit inferior of $A(t)$ as t tends to infinity, for small k, in terms of the zeros of the limiting function $\mu(A)$. The function f satisfies the fundamental assumption if it is independent of t, or is an almost periodic function in t given by an absolutely uniformly convergent Fourier series or tends uniformly to such a function.

F. Bohnenblust (Pasadena, Calif.).

Gomory, R., and Richmond, D. E. Boundaries for the limit cycle of van der Pol's equation. Quart. Appl. Math. 9, 205–209 (1951).

Upper and lower estimates for the maximal amplitude of the periodic solution of van der Pol's equation

$$\ddot{x}+\mu(x^2-1)\dot{x}+x=0$$

are obtained. The method consists in constructing in the phase plane two simple closed curves, an outer curve and an inner curve which form a ring containing the limit cycle of van der Pol's equation. These curves are obtained by dividing the x-axis into intervals, replacing the given differential equation in each interval by another appropriately chosen equation which is simple enough to be integrated explicitly. Solutions of these equations are then combined with the closed curves which limit the periodic solution of the given equation. As the authors point out, great accuracy can be obtained by a fine decomposition into intervals, but good results can be derived by a decomposition into three intervals.

F. Bohnenblust (Pasadena, Calif.).

Rubbert, Friedrich Karl. Erzwungene Pendelschwingungen endlicher Amplitude. Z. Physik 127, 72–84 (1950).

In order to explain the differences in the qualitative behavior of the solutions of Duffing's equation

$$\ddot{x}+n^2x=bx^3+a\cos mt,$$

the author applies a method similar to that used by Poincaré in perturbation problems. The initial conditions are taken to be $x(0)=\alpha$, α small, and $\dot{x}(0)=0$. The solution is expanded in a series in α after the substitution $\tau=nt/(1+\lambda\alpha^2)^{1/2}$. Only two terms are considered: $x=\alpha f(\tau)+\alpha^3 g(\tau)$, and higher terms in α are neglected. Two linear differential equations are obtained for the functions f and g. The parameter λ is so determined that the solutions for f and g remain bounded in time, λ appears as a root of a polynomial of third degree. The behavior of the resulting approximate solution is discussed at length. No convergence properties of a full expansion or estimates of error are considered.

F. Bohnenblust (Pasadena, Calif.).

Weidenhammer, F. Resonanzlösungen inhomogener Mathieuscher Systeme. Z. Angew. Math. Mech. 32, 154–156 (1952).

The system considered is

$$\omega^2 d^2 v_m/ds^2+\omega_m^2 v_m+\epsilon\cos s\sum_{n=1}^{N} F_{mn}v_n=\epsilon H_m\cos s$$

$(m=1, 2, \cdots, N)$, where $\epsilon<1$. A particular even solution of the equation is discussed qualitatively with special attention to the conditions for resonance. At non-resonance the solution is periodic of period 2π. At resonance it grows as a power of s.

E. Pinney (Berkeley, Calif.).

Popov, B. S. Sull'equazione di Bessel. Boll. Un. Mat. Ital. (3) 7, 17–19 (1952).

It is well known that Bessel functions are finite combinations of elementary functions if and only if the order is the half of an odd integer. The author points out that Bessel's differential equation is reducible under the same circumstances and exhibits the factorization explicitly.

A. Erdélyi (Pasadena, Calif.).

Sexl, Theodor. Zur Theorie der Laguerreschen Differentialgleichung. Acta Physica Austriaca 5, 449–460 (1952).

The author discusses the differential equation

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+\lambda y=0,$$

giving integral representations of significant solutions, expansions for large and small x. [There is no reference to the extensive literature on confluent hypergeometric functions which contains most of the results presented in this paper.]

A. Erdélyi (Pasadena, Calif.).

Wolfson, Kenneth G. On the spectrum of a boundary value problem with two singular endpoints. Amer. J. Math. 72, 713–719 (1950).

Let a positive $p(t)$ and a real-valued $q(t)$ be continuous functions of t, $-\infty<t<+\infty$, such that the differential equation $(px')'+(q+\lambda)x=0$ is of the "Grenzpunkt" type for both $t=-\infty$ and $t=+\infty$. Then $L\cdot x=-(px')'-qx$ is a self-adjoint operator on the Hilbert space $L^2(-\infty,+\infty)$ with the domain defined in a well-known manner. Similarly, imposing a fixed boundary condition:

$$x(c)\cos\alpha+x'(c)\sin\alpha=0$$

on $x=x(t)$ at a point c, $-\infty<c<+\infty$, we get a self-adjoint operator $L^a\cdot x$ (or $L_a\cdot x)=-(px')'-qx$ defined on the Hilbert space $L^2(-\infty,c)$ (or $L^2(c,+\infty)$). Again, let $N(a,b,\lambda_1,\lambda_2)$ be the number of eigen-values λ in the closed interval $\lambda_1\leqq\lambda\leqq\lambda_2$ of the Sturm-Liouville problem $(px')'+(q+\lambda)x=0$ belonging to the boundary conditions $x(a)=0$ and $x(b)=0$, and set $n(\lambda_1,\lambda_2)=\liminf N(a,b,\lambda_1,\lambda_2)$ where $a\to-\infty$ and $b\to+\infty$. In this paper the author has proved the following results: I) Denote by S the spectrum of L and by S' the derived set of S. Then λ belongs to S (or S') if and only if $n(\lambda-\epsilon,\lambda+\epsilon)\geqq1$ (or $=+\infty$) for every $\epsilon>0$. II) If $n(\lambda_1,\lambda_2)<+\infty$, then $n(\lambda_1,\lambda)$ is a non-decreasing step function of λ, $\lambda_1<\lambda<\lambda_2$, which is continuous from the left, and every jump of $n(\lambda_1,\lambda)$ has the value 1; moreover, in the open interval (λ_1,λ_2), the spectrum S of L consists of all discontinuity points of $n(\lambda_1,\lambda)$. III) Denote by S^* or S_* the derived set of the spectrum of L^a or L_a, respectively. Then we have $S'=S^*\cup S_*$. *K. Kodaira* (Princeton, N. J.).

Falk, Gottfried. Die Analogie zwischen Hamilton-Jacobi-Theorie und quantenmechanischem Eigenwertproblem. Z. Physik 131, 470–480 (1952).

Kahan, Théo, et Rideau, Guy. Sur un principe variationnel général en physique théorique. C. R. Acad. Sci. Paris 233, 849–852 (1951).

The variational principle for the inhomogeneous equation $L\psi=f$ is formulated and is applied to the binding energy, three-dimensional and one-dimensional scattering problems. *H. Feshbach* (Cambridge, Mass.).

Pastori, Maria. Integrazione tensoriale. Rend. Sem. Mat. Fis. Milano 21 (1950), 90–104 (1951).

The paper gives a review of some known results on the integration of invariant systems of differential equations in R_n and V_n as for instance rot $T=0$ and div $T=0$. The last section contains applications. *J. A. Schouten* (Epe).

Pucci, Carlo. Alcune limitazioni per gl'integrali delle equazioni differenziali a derivate parziali, lineari, del secondo ordine, di tipo ellittico-parabolico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 11, 334–339 (1951).

Let S be an open region with boundary D. Assume that at each point P of D there exists a line segment l which lies wholly in S except for point P. Let $u=u(x_1,\cdots,x_n)$ together with its first and second partial derivatives be continuous in S, have continuous first derivatives in $S+D$, and in S satisfy the equation

$$(1)\qquad \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}+cu=f.$$

Here a_{ij}, b_j, c, f are functions of $(x_1,\cdots,x_n)$ with $c=c(x_1,\cdots,x_n)<0$ in S. Assume further that on D the following relations hold

$$(2)\qquad \alpha(x_1,\cdots,x_n)\frac{du}{dl}+\beta(x_1,\cdots,x_n)u=\gamma(x_1,\cdots,x_n),\qquad \alpha\beta\leqq0,\quad \beta\neq0.$$

If for all real λ_i the quadratic form $\sum a_{ij}\lambda_i\lambda_j\geqq0$, the author shows that u satisfies the following inequality in $S+D$:

$$\min\left\{\min_S\left[\frac{f}{c}\right],\ \min_D\left[\frac{\gamma}{\beta}\right]\right\}\leqq u \leqq\max\left\{\max_S\left[\frac{f}{c}\right],\ \max_D\left[\frac{\gamma}{\beta}\right]\right\}.$$

Under certain additional conditions on the boundary D and the line segment l, the author shows if u is a solution of (1) in $S+D$ and at each point of D one has $du/dl=0$, then in $G\equiv S+D$ the following inequality holds

$$\min_G\left[\frac{f}{c}\right]\leqq u\leqq\max_G\left[\frac{f}{c}\right].$$

F. G. Dressel (Durham, N. C.).

***Brousse, Pierre. Étude d'équations aux dérivées partielles rencontrées dans la théorie des phénomènes de torsion.** Publ. Sci. Tech. Ministère de l'Air, no. 257, Paris, 1952. ii+76 pp. 600 francs.

The object of this monograph is to study the torsion in a shaft which is the solid of revolution formed by taking a circular cylinder of infinite length and cutting in it a groove of arbitrary form. The problem depends on solving the partial differential equation

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}=\frac{3}{y}\frac{\partial U}{\partial y},$$

where x is distance along the shaft, y distance from the axis of the shaft. The solution U has to vanish on the axis of the shaft and take a given constant value on the curve Σ in which the surface of the shaft is cut by an axial plane. One has thus to deal with a problem of Dirichlet of unusual form (a) because the domain D between $y=0$ and Σ, in which the solution has to be found, is of infinite extent, and (b) because the coefficient $3/y$ in the equation is infinite on the axis of x. In fact, to ensure uniqueness of the solution, one has to impose further restrictions on U; the author assumes U to be regular in D and bounded at infinity. Again, even if U exists and is unique, one cannot assert a priori that it has a normal derivative $\partial U/\partial n$ on the boundary of D. Moreover, even if this normal derivative does exist, it could happen that $y^{-2}\partial U/\partial n$ is infinite on $y=0$, which is not acceptable physically. These are the fundamental questions solved in this monograph. Some of the results have already been announced in C. R. Acad. Sci. Paris 230, 713–714 (1950); these Rev. 11, 521.

E. T. Copson (St. Andrews).

Brousse, Pierre. Sur une équation de la mécanique des milieux continus. C. R. Acad. Sci. Paris 234, 2146–2148 (1952).

This contains a statement without proofs of a group of results concerning the solutions of the partial differential equation

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}-\frac{k}{y}\frac{\partial U}{\partial y}=0$$

where k is a positive constant. *E. T. Copson.*

Datzeff, Assène. Sur le problème linéaire de Stefan. II. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 271–325 (1950). (French. Bulgarian summary)

[For part I see same Annuaire **45**, 321–352 (1949); these Rev. **12**, 504.] The problem of Stefan is concerned with the distribution of temperature in the liquid and solid states (A_1 and A_2) of a homogeneous medium. Consider, for example, finite depths of water and ice in contact along a plane perpendicular to the x-axis. The bodies A_1 and A_2 extend indefinitely in directions perpendicular to the x-axis. On the plane where the two phases touch, the temperature is constant and equal to the temperature of fusion ϕ_0. If the initial distributions of temperatures in A_1 and A_2 are functions of x alone, then the problem is one in a single space variable and A_1 and A_2 may be thought of as bars of finite length lying along the x-axis, whose lateral surfaces are impermeable to heat.

If $u_1(x, t)$, $u_2(x, t)$ are the temperatures in A_1 and A_2, they must satisfy the equations

$$(1)\qquad a_i^2\frac{\partial^2 u_i}{\partial x^2}=\frac{\partial u_i}{\partial t}\quad (i=1, 2),$$

the initial conditions

$$(2)\qquad \begin{aligned} u_1(x, t_0)&=\Phi_1(x)\quad (x'<x<x_0),\\ u_2(x, t_0)&=\Phi_2(x)\quad (x_0<x<x''),\end{aligned}$$

and the boundary conditions

$$(3)\qquad u_1(x', t)=\phi_1(t),\quad u_2(x'', t)=\phi_2(t)\quad (t>t_0),$$

where Φ_1, Φ_2, ϕ_1, ϕ_2 are arbitrary bounded integrable functions. At the variable point of contact $x_0=s(t)$ of A_1 and A_2 the condition of Stefan holds:

$$(4)\qquad \frac{ds}{dt}=e\left[k_1\frac{\partial u_1}{\partial x}-k_2\frac{\partial u_2}{\partial x}\right]_{x=s(t)},\quad e=1/\rho\sigma,$$

where ρ is the density of both A_1 and A_2 and σ is the heat of fusion. The problem is to find the functions $u_1(x, t)$, $u_2(x, t)$ and $s(t)$ which satisfy the conditions (1), (2), (3), and (4).

The author first solves an auxiliary problem in which he considers a single bar of finite length lying along the x-axis between the points $O_s(s)$ and $O'(x')$. The abscissa s of the point O_s varies according to the law $x=s(t)$ while the temperature there remains constant ϕ_0. The temperature at the fixed point O' is a given function $\phi(t)$ of time. Explicitly, this problem is as follows: find the temperature $u(x, t)$ in the bar A satisfying the equation (5) $a^2\partial^2u/\partial x^2=\partial u/\partial t$, the initial conditions (6) $u(x, t_0)=\Phi(x)$, $(x_0<x<x')$, and the boundary conditions

$$(7)\qquad u(s(t), t)=\phi_0,\quad u(x', t)=\phi(t)\quad (t_0<t<t_0+T).$$

The method of solution consists of dividing the interval T into n parts $\Delta t_i=t_{i+1}-t_i$ $(i=1, 2, \cdots, n)$ and solving the problem represented by (5), (6), and (7) with $s(t)$ replaced by the step-function s_i which is constant within each of the subdivisions Δt_i. This leads to a set of functions

$$u_i(x, t)\equiv u_{in}(x, t)\quad (t_i<t<t_{i+1},\ x_i<x<x')\quad (i=0, 1, 2, \cdots)$$

with initial temperatures

$$u_i(x, t_i)=u_{i-1}(x, t_i)=\Phi_i(x)\quad (x_i<x<x')$$

and boundary conditions $u_i(x_i, t)=\phi_0$, $u_i(x', t)=\phi(t)$. It is then shown that the limit as $n\rightarrow\infty$ of the sequence $u_{in}(x, t)$ exists and satisfies equations (5), (6) and (7).

Returning to the problem of Stefan, the unknown function $s(t)$, giving the law of motion of the point of contact O_s between the two bars A_1 and A_2, is replaced by a step-function $s_i=s_{in}(t)$ $(s_i=s(t_i))$. The functions u_{1i}, u_{2i} become known when the law of motion $s_i(t)$ of O_s is known, and this is determined from the condition

$$(8)\qquad \frac{ds}{dt}=e\left[k_1\frac{\partial u_{1i}}{\partial x}-k_2\frac{\partial u_{2i}}{\partial x}\right]_{x=s(t_i)}$$

which replaces the condition (4). The functions u_{1i} and u_{2i} for $i=0$ are found by the method of the auxiliary problem. Substituting these into (8) and integrating we get $s_1=s(t_1)$. Putting this value of s_1 into u_{1i} and u_{2i} for $i=1$ and again using the method of the auxiliary problem, these functions are determined. Then $s_2=s(t_2)$ is found from (8). This is repeated to find the sequences u_{1i}, u_{2i}, s_i $(i=0, 1, 2, \cdots, n)$.

If now $n\rightarrow\infty$ and $\Delta t\rightarrow 0$ simultaneously, condition (8) is transformed into condition (4). In this way are found the functions $s(t)$, $u_1(x, t)$ and $u_2(x, t)$ which satisfy (1), (2), (3) and (4) and represent the solution of the problem. It is shown that this solution exists and is unique.

If the boundary conditions (3) are replaced by

$$(3')\qquad u_1(x', t)=\phi_1(t),\quad \partial u_2(x'', t)/\partial x=\phi_2(t),$$

or by

$$(3'')\qquad \partial u_1(x', t)/\partial x=\phi(t),\quad \partial u_2(x'', t)/\partial x=\phi_2(t),$$

it is shown that the problems so obtained can be solved by a method analogous to that outlined above. *C. G. Maple.*

John, Fritz. On integration of parabolic equations by difference methods. I. Linear and quasi-linear equations for the infinite interval. Comm. Pure Appl. Math. **5**, 155–211 (1952).

In the infinite strip R: $0\leqq t\leqq T$, $-\infty<x<\infty$ introduce a family of rectangular lattices Σ_h with mesh (h, k). Here h and k vary in such a way that $k/h^2=\lambda$ is a fixed constant. For (x, t) in the lattice Σ_h let $u(x, t, h)$ be a solution of the recursion system

$$(1)\qquad \begin{aligned} u(x, t+k, h)&=\sum_{r=-\infty}^{\infty}c^r(x, t, h)u(x+rh, t, h)+kd(x, t),\\ u(x, 0, h)&=f(x).\end{aligned}$$

The c^r are chosen so that the above difference system will, for $h\rightarrow 0$, go over formally into the differential system

$$(2)\qquad \begin{aligned} \frac{\partial u}{\partial t}&=a_0(x, t)\frac{\partial^2 u}{\partial x^2}+2a_1(x, t)\frac{\partial u}{\partial x}+a_2(x, t)u+d(x, t),\\ &\qquad\qquad 0<t<T,\\ u(x, 0)&=f(x).\end{aligned}$$

We now state some of the author's results relative to solutions of systems of type (1) leading to a solution of system (2). Assume $a_0>0$, $\partial a_0/\partial x$, $\partial^2 a_0/\partial x^2$, a_1, $\partial a_1/\partial x$, a_2, d are uniformly continuous and bounded in R. If $f(x)$ is bounded for all x and Riemann integrable over every finite interval, the author gives restriction on the c^r sufficient to insure that $\lim_{h\rightarrow 0}u(x, t, h)=U(x, t)$ exists. Here $u(x, t, h)$ is a solution of system (1). If, in addition, $\partial a_2/\partial x$, $\partial d/\partial x$ are uniformly continuous and bounded in R, then $U(x, t)$ will be a solution of (2) and $U(x, 0)=f(x)$ at every point of continuity of the given function $f(x)$. The author also obtains results along this same line for the quasi-linear differential equation

$$\frac{\partial u}{\partial t}=a_0\frac{\partial^2 u}{\partial x^2}+2a_1\frac{\partial u}{\partial x}+d(x, t, u).$$

F. G. Dressel (Durham, N. C.).

Feller, William. The parabolic differential equations and the associated semi-groups of transformations. Ann. of Math. (2) 55, 468–519 (1952).

This paper gives an exhaustive study of the pair of diffusion equations

(1) $u_t(t,x)=a(x)u_{xx}(t,x)+b(x)u_x(t,x)$ in $C(r_1,r_2)$

and

(2) $v_t(t,y)=\{(a(y)v(t,y))_y-b(y)v(t,y)\}_y$ in $L_1(r_1,r_2)$,

where $-\infty\leqq r_1\leqq r_2\leqq\infty$. The method of attack lies in a systematic use of the theory of one-parameter semigroups due to E. Hille and the reviewer, viz. in the study of the resolvent $(\lambda I-ad^2/dx^2-bd/dx)^{-1}$ and its adjoint. The completeness of the results are due to "the lucky circumstance" that these studies are of ordinary differential equations.

It is assumed that $a'(x)$ and $b(x)$ are continuous in the open interval (r_1,r_2) and, furthermore, $a(x)>0$. By virtue of Hille's function $W(x)=\exp(-\int_{x_0}^x b(s)a^{-1}(s)ds)$, the nature of the boundary r_2 (and similarly of r_1) is classified: i) "regular" if $W(x)\varepsilon L_1(x_0,r_2)$ and $a^{-1}(x)W^{-1}(x)\varepsilon L_1(x_0,r_2)$; ii) "exit" if $a^{-1}W\bar{\varepsilon}L_1(x_0,r_2)$ and $W(x)\int_{x_0}^x a^{-1}(s)W^{-1}(s)ds\,\varepsilon\,L_1(x_0,r_2)$; iii) "entrance" if $a^{-1}W^{-1}\varepsilon L_1(x_0,r_2)$ and

$$a^{-1}(x)W^{-1}(x)\int_{x_0}^{x}W(s)ds\,\varepsilon\,L_1(x_0,r_2);$$

iv) "natural" in all other cases. When the boundary r_j is regular, the boundary conditions of the type

$$q_j\lim_{x\to r_j}u(t,x)+p_j(-1)^j\lim_{x\to r_j}W^{-1}(x)u_x(t,x)=0$$

are in one-one correspondence to the common fundamental solutions of (1) and (2). If the two boundaries are natural, the initial value problem both for (1) and (2) is uniquely determined and the solutions are generated by the unique common fundamental solution, no boundary conditions being imposed. If r_1 be a natural and r_2 an exit boundary, the initial value problem for (1) has infinitely many solutions but that for (2) is uniquely determined.

The most remarkable result is the discovery that the corresponding adjoint of (1) is no longer the differential equation (2). It is, for example, of the form

(3) $v_t(t,x)=\{(a(x)v(t,x))_x-b(x)v(t,x)\}_x+r\sigma^{-1}V_2(t)p'(t)$,

where

$$V_1'(t)=p_1\sigma^{-1}V_2(t),$$
$$V_2'(t)=-p_2\sigma^{-1}V_2(t)-\lim_{x\to r_2}\{(a(x)v(t,x))_x-b(x)v(t,x)\}_x.$$

Thus an essentially new boundary condition is introduced: We interpret $v(t,x)$ as the mass density at the time moment t in the interior of (r_1,r_2) and $V_j(t)$ as mass at r_j. Then the mass flows out of the interior at r_2 but not at r_1. The mass concentrates at r_2 from which it flows out at the constant rate p_2/σ. The outflowing mass goes in part to r_1, in part to the interior of (r_1,r_2) and in part it disappears, the three parts being in the ratios $p_1:r:(p_2-p_1-r)$. Such "true" Fokker-Planck equations (3) are obtained by considering the adjoint of the resolvent $(\lambda I-ad^2/dx^2-bd/dx)^{-1}$ in which the boundary conditions are attached. *K. Yosida.*

Giuliano, Landolino. Sull'unicità della soluzione per una classe di equazioni differenziali alle derivate parziali, paraboliche, non lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 260–265 (1952).

Let $f(x,t,u,p,r)$ be a monotone increasing function of r, and have the additional property that if $u_1\geqq u_2$

(1) $f(x,t,u_1,p,r)-f(x,t,u_2,p,r)\leqq h(t,u_1-u_2)$.

Here $h(t,\theta)$ is such a function that $\theta\equiv 0$ is the only solution of $d\theta/dt=h(t,\theta)$ for which $\lim_{t\to 0}\theta(t)=\lim_{t\to 0}\theta'(t)=0$. It is proved that there exists at most one solution $u=u(x,t)$ of the nonlinear parabolic equation $u_t=f(x,t,u,u_x,u_{xx})$ in the rectangle $0<x<1$, $0<t<T$, if the values of u are prescribed on the two vertical and lower horizontal sides of the rectangle and if u, u_t, u_x, u_{xx} are continuous for $0\leqq x\leqq t$, $0\leqq t\leqq T$. This generalizes a result of Westphal [Math. Z. 51, 690–695 (1949); these Rev. 11, 252]. In place of condition (1), Westphal imposed a Lipschitz condition on f with respect to the argument u. The author states that his results can be extended to equations of the type

$$u_t=f(x_1,\cdots,x_n;t;u;u_{x_1},\cdots,u_{x_n};u_{x_1x_1},\cdots,u_{x_nx_n})$$

and also to systems of equations of this type.

F. G. Dressel (Durham, N. C.).

Petrašen', M. I. On semiclassical methods of solution of the wave equation. Leningrad. Gosudarstv. Univ. Učenye Zapiski 120, Ser. Fiz. Nauk 7, 59–78 (1949). (Russian)

The one-dimensional wave equation is studied by means of changes in the independent variable and by replacing the dependent variable by a new dependent variable multiplied by a function of the independent variable. Reference is made to the BWK method and many examples illustrate the discussion. *N. Levinson* (Cambridge, Mass.).

Visvanathan, S. A simple method of solving D'Alembert's equation $\Box^2\phi=-\lambda$. Math. Student 18 (1950), 27–30 (1951).

The author derives the usual retarded or advanced potential solution of

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\lambda(x,y,z,t),$$

viz.

$$4\pi\phi=\int\frac{[\lambda]}{r}dv+\int\left\{\frac{1}{r}\left[\frac{\partial\phi}{\partial n}\right]-[\phi]\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\mp\frac{\cos\psi}{c r}\left[\frac{\partial\phi}{\partial t}\right]\right\}ds$$

by the change of variable $x'=x$, $y'=y$, $z'=z$, $t'=t\mp r/c$ and an ingenious use of Green's transformation.

E. T. Copson (St. Andrews).

Storchi, Edoardo. Condizioni al contorno per le equazioni alle derivate parziali lineari del terzo ordine a coefficienti constanti. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 208–218 (1950).

Let the domain D in the (x,y)-plane be bounded by a curve s, which is intersected in at most 2 points by parallels to the axes. Let u be a solution of a third order equation with constant coefficients of the form

$$Au_{xxx}+Cu_{xyy}+eu_{xx}+gu_{yy}+hu_x+iu_y=0,$$

where e and g are positive. Then u is uniquely determined in D by the values of u on s and the values of the normal derivative of u on the "left half" of s. The proof follows from a suitable quadratic integral identity. *F. John.*

★**Pleijel, Åke. On Green's functions for elastic plates with clamped, supported and free edges.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 413–437. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

In his paper on eigenvalue distributions, H. Weyl [Rend. Circ. Mat. Palermo 39, 1–50 (1915)] considered various three-dimensional differential problems, and in a series of

lemmas he derived estimates for the Green's functions connected with them. For example, for the Green's function $G(p,q)=(4\pi r_{pq})^{-1}-\gamma(p,q)$, for the Dirichlet problem for Laplace's equation for the three-dimensional region V, Weyl showed that the "regular part" γ satisfies

$$\gamma(p,q)=O(1/R(p,q)),$$

where R denotes the "light distance" between the points p and q of V after one reflection at the boundary S of V, i.e. $R(p,q)=\min_{s\in S}(r_{ps}+r_{qs})$. In the present paper the author considers analogous questions concerning the Green's functions for three boundary value problems for the two-dimensional biharmonic equation $\Delta\Delta u=0$ in a plane domain V with a smooth boundary, with boundary conditions corresponding to those of a clamped plate, a simply supported plate, and a plate with free edges. The Green's functions for these three problems are constructed with the help of integral equations, after modifying the boundary conditions slightly in order to be able to apply the Fredholm theory. It is shown that if $\gamma(p,q)$ denotes the regular part of Green's function for any of these problems, then

$$\gamma(p,q)=O(1),$$
$$D\gamma(p,q)=O(1),$$
$$D^2\gamma(p,q)=O(|\log R(p,q)|),$$
$$D^{k+2}\gamma(p,q)=O(R^{-k}(p,q)),\quad k>0,$$

where $D=\partial/\partial x$ or $\partial/\partial y$, and $R(p,q)$ is the "light distance" of the points p and q of V from the boundary of V. It is also shown how to construct the requisite Green's functions by using Weyl's method. *J. B. Diaz.*

***Pleijel, Åke. Green's functions and asymptotic distribution of eigenvalues and eigenfunctions.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 439–454. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

The asymptotic distribution of the eigenvalues and eigenfunctions of membrane problems and similar differential problems was first studied by H. Weyl [Nachr. Ges. Wiss. Göttingen **1911**, 110–117; Rend. Circ. Mat. Palermo **39**, 1–50 (1915)] and R. Courant [Math. Z. **15** (1922), 195–200; see Courant and Hilbert, Methoden der mathematischen Physik, Bd. I, Springer, Berlin, 1937], the eigenvalue distributions for an arbitrary region being deduced from those for a rectangle, or, in three dimensions, for a parallelepiped. Problems concerning elastic plates are more difficult since the asymptotic law for the eigenvalues for a rectangle cannot be explicitly computed. R. Courant, in the paper cited above, solved the problem for plates with the boundary condition $u=0$, $\partial u/\partial n=0$, from knowing the spectrum of a circular plate. The first theorems of a similar type for eigenfunctions, instead of eigenvalues, were given by T. Carleman [C. R. 8ième Congrès des Mathématiciens Scandinaves, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44; Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **88**, 119–132 (1936)] who at the same time found a new proof of Weyl's law for the eigenvalues. The present paper begins with a succinct exposition of Carleman's idea of reducing the problem of eigenvalue-eigenfunction distribution to an investigation of the behavior of (finding estimates for) the compensating part ("regular part") of Green's function. Various methods (maximum-minimum principle for solutions of certain equations, Neumann series expansion, variational) for obtaining the required estimates of the compensating part of Green's function are then presented, and the eigenvalue-eigenfunction distribution for various problems (clamped, simply supported, free) for the elastic plate are deduced [Pleijel, Comm. Pure Appl. Math. **3**, 1–10 (1950); these Rev. **12**, 265]. The paper concludes with a discussion of various related problems, in particular, a problem containing the eigenvalue parameter in the boundary conditions. *J. B. Diaz* (College Park, Md.).

Difference Equations, Special Functional Equations

Duff, G. F. D. F-equation Fourier transforms. Canadian J. Math. **4**, 248–256 (1952).

Certain solutions of the F-equation,

$$dF(z,\alpha)/dz=F(z,\alpha+1),$$

may be represented as power series

$$F(z,\alpha)=\sum_{n=0}^{\infty}f(\alpha+n)z^n/n!$$

and others as contour integrals

$$F(z,\alpha)=\int_C \exp(2\pi i\alpha s+ze^{2\pi is})g(s)\,ds.$$

If a solution $F(z,\alpha)$ possesses both representations and C is the real axis, then $f(\alpha)$ and $g(s)$ are Fourier transforms of each other. A conjugate solution $G(z,\alpha)$ may be obtained by replacing $f(\alpha)$, $g(s)$ by $g(\alpha)$, $f(-s)$. The author employs standard techniques (convolution, Poisson's summation formula) to obtain certain relationships for solutions of the F-equation. *A. Erdélyi* (Pasadena, Calif.).

Pastidès, Nicolas. Sur une généralisation de l'équation fonctionnelle de Schroeder-Koenigs. C. R. Acad. Sci. Paris **234**, 2417–2418 (1952).

The equation in question is

$$(*)\quad F[f_1(x_1,\cdots,x_n),\cdots,f_n(x_1,\cdots,x_n)]=sF(x_1,\cdots,x_n)$$

where $f_1,\cdots,f_n$ are given, s=constant, and F is to be determined. The functions f_i $(i=1,\cdots,n)$ are assumed to be analytic in a neighborhood of the origin, to be zero at the origin, and to have a system of inverses. One can then take $f_i(x_1,\cdots,x_n)=\sum_{j=1}^{n}\alpha_j^i x_j+\sum\alpha^i_{r_1\cdots r_n}x_1^{r_1}\cdots x_n^{r_n}$ where the second sum contains the nonlinear terms. Set up the iteration process:

$$f_i^1(x_1,\cdots,x_n)=f_i(x_1,\cdots,x_n);$$
$$f_i^m(x_1,\cdots,x_n)=f_i^{m-1}(f_1,\cdots,f_n),\quad m=2,3,\cdots.$$

It is shown that if for some value i the sequence $f_i^p(x_1,\cdots,x_n)$ $(p=1,2,\cdots)$ is a normal family in a domain D containing the origin in its interior, then equation (*) has an analytic solution F in the neighborhood of the origin, corresponding to the value $s=a_i^i$. (The family $\varphi_{i,p}(x_1,\cdots,x_n)=p^{-1}\sum_{j=1}^{p}f_i^j/s^j$ is normal in every closed domain of D, so it possesses a subsequence that converges uniformly, and the limit function is the desired solution F.) *I. M. Sheffer.*

Bencivenga, Ulderico. Su di un metodo per la ricerca di alcune funzioni. Ricerca, Napoli **1**, no. 4, 24–33 (1950); **2**, no. 1, 28–32; no. 2, 25–32; no. 3–4, 22–26 (1951).

This study concerns the functional equation

$$f[\phi(x,y,z,\cdots)]=F[f(x),f(y),f(z),\cdots].$$

ϕ and F are symmetric functions, $f(x)$ the unknown function —all assumed to possess derivatives of all orders involved

in the developments. The "theorem on the factor of symmetry" states: Functions F and ϕ being given, if

$$[\partial F/\partial f(x)][\partial\phi/\partial y][\partial\phi/\partial z]\cdots$$

can be rendered symmetric through multiplication by a function S of x, there exists a solution $f(x)$ which equals $\int S\,dx$, provided the product equals

$$[df/d\phi][\partial\phi/\partial x][\partial\phi/\partial y][\partial\phi/\partial z]\cdots$$

identically. Applications studied are:

$$\phi=x+y,$$

$$F=\varphi(X)\psi(Y)+\varphi(Y)\psi(X)+\mu(X)\nu(Y)+\mu(Y)\nu(X)+\cdots$$

where X indicates $f(x)$ and Y is $f(y)$. Subcases treated in greater detail are

$$F=\varphi(X)\psi(Y)+\varphi(Y)\psi(X)$$

and

$$F=\varphi(X)\psi(Y)+\varphi(Y)\psi(X)+\mu(X)\mu(Y)$$

$$F=\frac{\varphi(X)\psi(Y)+\varphi(Y)\psi(X)}{\mu(X)\nu(Y)+\mu(Y)\nu(X)}.$$

E. S. Allen (Ames, Iowa).

Bajraktarević, Mahmud. Sur la convergence de la suite définie par la formule $x_{n+1}=f(x_n)$, $n=0, 1, 2, \cdots$. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 6, 201–209 (1951). (Serbo-Croatian. French summary)

It is shown that the sequence $x_{n+1}=f(x_n)$, $n=0, 1, 2, \cdots$, converges to a if, in some interval about a containing x_0, $f(x)$ is continuous and $|f(x)-a|<|x-a|$ for $x\neq a$. The author gives a few more such results on the convergence of the sequence (x_n). *M. Golomb* (Lafayette, Ind.).

Integral Equations

Rabinovič, Yu. L. On the continuous dependence upon a parameter of the characteristic values of linear integral equations. Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 172–174 (1952). (Russian)

The author proves the following result. Let $K(x, y; \mu)$ be defined for $x, y \varepsilon B$, $\mu\varepsilon G$, where B is a measurable set in R^m, and G is a domain of the complex plane, and suppose that K is continuous in μ for almost all $(x, y)\varepsilon B\times B$ and measurable in (x, y) for all $\mu\varepsilon G$, and that $|K(x, y; \mu)|<\varphi(x)$, where φ is summable over B. Then the characteristic values of the kernel $K(x, y; \mu)$ depend continuously on the parameter μ. The result is then extended to the case where the kernel itself does not have the above properties, but one of its iterates does. [Reviewer's note. The author's proof requires modification to allow for the possibility that $K(x, x; \mu)$ is not a measurable function of x; this causes no real difficulty.] *F. Smithies* (Cambridge, England).

Chang, Shih-Hsun. A generalization of a theorem of Hille and Tamarkin with applications. Proc. London Math. Soc. (3) 2, 22–29 (1952).

Hille and Tamarkin [Acta Math. 57, 1–76 (1931), p. 46] proved a theorem of which the following result quoted by the author is a rather special result. If the kernel $K(x, y)$, normally unsymmetric, has partial derivatives with respect to x of order $\leqq p$ and if the pth derivative is L_2 over the fundamental square, then $|\mu_n[K]|^{-1}=o(n^{-p-\frac{1}{2}})$ holds for the nth characteristic value. By an adaptation of an argument due to M. Krein for symmetric kernels [Mat. Sbornik 2(44), 725–732 (1937)], the author proves that the same inequality holds for the singular values in the sense of E. Schmidt and this gives him an alternate proof of the theorem quoted above. He also proves that such a kernel can be factored in the sense of functions of composition into at least $2p$ kernels in L_2. *E. Hille* (Paris).

Heinhold, J. Zur Konstruktion involutorischer Kerne. Arch. Math. 3, 15–23 (1952).

$K(t, \tau)$ is a reciprocal kernel if the solution of the integral equation of the first kind $G(t)=\int_0^\infty K(t, \tau)F(\tau)d\tau$ is

$$F(t)=\int_0^\infty K(t, \tau)G(\tau)d\tau.$$

The author constructs a class of reciprocal kernels as the inverse Laplace transforms of $\phi(s)\exp[-\tau\psi(s)]$ where $\phi(s)$ is a suitable analytic function of s, ϕ^{-1} is the inverse function and $\psi(s)=\phi^{-1}(1/\phi(s))$. [See also Parodi, C. R. Acad. Sci. Paris 226, 1877–1878; 227, 810–812 (1948); Bull. Sci. Math. (2) 72, 66–68 (1948); these Rev. 10, 36, 370, 459. Parodi has also considered the corresponding integral equation of the second kind [J. Math. Pures Appl. (9) 28, 35–62 (1949); these Rev. 10, 715] but he gives no rigorous proofs.] *A. Erdélyi* (Pasadena, Calif.).

Mönnig, Paul. Über Integralgleichungen mit unsymmetrischem Polynomkern bei längs der Hauptdiagonale sich änderndem Bildungsgesetz. Monatsh. Math. 56, 1–15 (1952).

The resolvent kernel of a Fredholm kernel of the special form

$$K(x, \xi)=\sum_1^n\phi_i(x)\psi_i(\xi)\quad(x\leqq\xi),\quad K(x, \xi)=\sum_1^n\bar\phi_i(x)\bar\psi_i(\xi)\quad(x>\xi),$$

is expressed in terms of ϕ_i, ψ_i, $\bar\phi_i$, $\bar\psi_i$ and other functions Φ_i, Ψ_i, $\bar\Phi_i$, $\bar\Psi_i$; $\Phi_i=\Phi_i(x, \lambda)$, for example, is the solution of the Volterra equation $\Phi_i(x, \lambda)-\lambda\int_a^x V(x, \xi)\Phi_i(\xi, \lambda)d\xi=\phi_i(x)$, where $V(x, \xi)=\sum_1^n\bar\phi_i(x)\bar\psi_i(\xi)-\sum_1^n\phi_i(x)\psi_i(\xi)$. The formulae are used to derive standard expressions for the solution of boundary-value problems for ordinary linear differential equations. Conversely, with certain restrictions, it is shown that a Fredholm equation with kernel $K(x, \xi)$ of the above special type may be converted into a boundary-value problem. *G. E. H. Reuter* (Manchester).

Fenyő, István. On a class of integral equations and its practical applications. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 120–130 (1951). (Hungarian)

The author considers integral equations whose kernels satisfy a relation of the form

$$\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y}=\sum_{i=1}^N a_i(x)b_i(x).$$

He mentions several physical problems leading to such equations. The principal result is that the proper functions of such an equation must satisfy certain boundary conditions. Thus, for example, if K is symmetric and $\partial K/\partial x+\partial K/\partial y=0$, and if $\varphi(x)=\lambda\int_0^1 K(x, y)\varphi(y)dy$, then $\varphi(0)=\pm\varphi(1)$. *P. R. Halmos* (Chicago, Ill.).

Citlanadze, È. S. On integral equations of Lichtenstein type. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 359–364 (1947). (Russian)

The author discusses Lichtenstein's non-linear integral equation

$$L\varphi(s)\equiv\sum_{n=1}^{\infty}\int_0^1\cdots\int_0^1 K_n(s,t_1,\cdots,t_n)\times\varphi(t_1)\cdots\varphi(t_n)dt_1\cdots dt_n=\lambda\varphi(s),$$

where each $K_n(s,t_1,\cdots,t_n)$ is a symmetric continuous function of $(s,t_1,\cdots,t_n)$ in the $(n+1)$-dimensional unit cube, and

$$\sum_{n=1}^{\infty}\left\{\max_{0\leqq s\leqq 1}\left(\int_0^1\cdots\int_0^1 K_n^2(s,t_1,\cdots,t_n)dt_1\cdots dt_n\right)\right\}<\infty.$$

Using the fact that the operator L is generated by the Fréchet differential of the weakly continuous functional

$$F(\varphi)\equiv\sum_{n=2}^{\infty}\frac{1}{n}\int_0^1\cdots\int_0^1 K_{n-1}(t_1,\cdots,t_n)\varphi(t_1)\cdots\varphi(t_n)dt_1\cdots dt_n,$$

and is therefore a completely continuous operator in Hilbert space, he shows that the equation (1) has at least one solution $\varphi(s)$ such that $\|\varphi\|=1$. If the functional $F(\varphi)$ is even and positive, there is an infinite sequence of characteristic values and corresponding characteristic functions.

F. Smithies (Cambridge, England).

Hahubia, G. P. On the theory of the resolvent for a functional equation of Günther's type. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 571–576 (1947). (Russian)

The author extends the theory of principal kernels and principal functions of integral equations to Günther's generalization of the Fredholm equation involving Stieltjes integrals [N. Günther, Trav. Inst. Phys.-Math. Stekloff **1**, 1–494 (1932)]. No essentially new methods are involved.

F. Smithies (Cambridge, England).

Goodman, Leo A. A probabilistic approach to a system of integral equations. Proc. Amer. Math. Soc. **3**, 505–507 (1952).

The following system of integral equations arises in statistical work:

$$G_n(y)=\int_0^y\frac{e^{-\omega}\omega^{n-1}}{(n\alpha)}d\omega=\int_0^{g(y)}H_{n-1}(g(y)-z;g)dH_1(z;g),\qquad n=1,2,\cdots,$$

where $g(x)\geqq 0$ is an increasing continuous function of $x\geqq 0$, $H_0(x;g)=1$, and $H_n(x;g)=\int_0^x H_{n-1}(x-z;g)dH_1(z;g)$. The author shows, using probabilistic methods, that the only functions satisfying this system of equations for fixed α, $0\leqq\alpha\leqq 2$, are $g(x)=cx$, where c is a constant.

J. L. Snell (Princeton, N. J.).

Jaeckel, K. Über die Eigenlösungen gewisser Integralgleichungen der Potentialtheorie. J. Reine Angew. Math. **189**, 141–149 (1951).

The author considers the equations:

a) $$\int_{-1}^1\varphi(\xi)[1-\xi^2]^{-1/2}\ln|\xi-x|d\xi=\lambda\varphi(x);$$

b) $$\frac{d^2}{dx^2}\int_{-1}^1\varphi(\xi)[1-\xi^2]^{1/2}\ln|\xi-x|d\xi=\lambda\varphi(x).$$

It is shown that the characteristic solutions are polynomials satisfying, respectively,

a) $D_x\varphi(x)+\mu^2\varphi(x)=0$; b) $d_x\varphi(x)+\mu^2\varphi(x)=0$,

where

$$D_x=-x\partial/\partial x+(1-x^2)\partial^2/\partial x^2,$$
$$d_x\varphi=(1-x^2)^{-1/2}D_x[\varphi(1-x^2)^{1/2}].$$

The change of variables $x=-\cos\vartheta$ leads easily to the known results: a) $\varphi_n=\cos n\vartheta$, $\lambda_0=-\pi\ln 2$, $\lambda_n=-\pi/n$; b) $\varphi_n=\sin n\vartheta/\sin\vartheta$, $\lambda_n=\pi n$. Application is made to a problem from airfoil theory. Finally, it is shown how the method may be extended to include the corresponding pair of equations from potential theory in three dimensions.

J. V. Wehausen (Providence, R. I.).

Ramakrishnan, Alladi. On an integral equation of Chandrasekhar and Münch. Astrophys. J. **115**, 141–144 (1952).

It is shown that the integral equation recently derived by Chandrasekhar and Münch [Astrophys. J. **112**, 380–392 (1950); these Rev. **12**, 644] to describe the fluctuations in brightness of the Milky Way is equivalent to the following problem in Markovian processes: "Given (1) that there is a deterministic contribution of amount $\beta d\tau$ from $d\tau$ at $t=\tau$ to the intensity u when the system extends to a distance $t=\xi$; (2) that the probability per unit t that an intensity of magnitude u drops to an interval lying between uq and $u(q+dq)$ is $\psi(q)dq$; given these, what is the frequency function $g(u,\xi)$ governing the intensity u?"

S. Chandrasekhar (Williams Bay, Wis.).

Guy, Roland. Existence de solutions pour des systèmes d'équations opératorielles intégrales à limite variable. C. R. Acad. Sci. Paris **234**, 918–920 (1952).

The theory developed in a previous note [same C. R. **233**, 288–290 (1951); these Rev. **13**, 194] is extended to a system with infinitely many unknowns. *C. C. Torrance.*

Functional Analysis, Ergodic Theory

Walsh, Michael John. The paracompactness of the *CW*-complex and gradient mappings in locally convex spaces. Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. ii+1+i pp.

Silverman, Robert Jerome. Invariant extensions of linear operators. Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. ii+2+i pp.

Bartle, Robert G., and Graves, Lawrence M. Mappings between function spaces. Trans. Amer. Math. Soc. **72**, 400–413 (1952).

$\mathfrak{U}$ and $\mathfrak{B}$ are Banach spaces, and $\mathfrak{T}$ is a Hausdorff space. $\mathfrak{X}$ and $\mathfrak{Y}$ are the spaces of continuous mappings of $\mathfrak{T}$ into $\mathfrak{U}$ and $\mathfrak{B}$ respectively, and Ω is the space of linear continuous mappings of $\mathfrak{U}$ into $\mathfrak{B}$. The authors consider functions K on $\mathfrak{T}$ to Ω which are bounded on $\mathfrak{T}$ and continuous in the strong topology of Ω. Given such a function, they define a linear continuous mapping κ of $\mathfrak{X}$ into $\mathfrak{Y}$ by the rule that the value taken by $\kappa(x)$ at t (denoted by $\kappa(x|t)$) is $K(t)x(t)$. If κ is given, K is defined by $K(t)u=\kappa(x_u|t)$, where $x_u(t)=u$ for all t. If we start with K and define κ, and then define K in

terms of κ, we get back to where we started. If, on the other hand, we start with κ, we get back to where we started if and only if $\kappa(x|t_0)$ is independent of the values taken by $x(t)$ at points other than t_0. If κ is in addition completely continuous, then the associated K is continuous with the uniform topology in Ω, and $K(t)$ is completely continuous for each t. The authors apply these and allied results to the solution of nonlinear and other equations.

A. F. Ruston (London).

Harazov, D. F. Application of the method of successive approximations to the solution of some functional equations. Soobščeniya Akad. Nauk Gruzin. SSR. **12**, 3–9 (1951). (Russian)

Let H_k, $k=0, 1, \cdots$, be a sequence of linear transformations of a Banach space into itself such that $\sum\lambda^k\|H_k\|$ has radius of convergence $\rho>0$. Let r_0 be the (unique) root of $\sum r^k\|H_k\|-1=0$ in $[0, \rho]$ if one exists, otherwise $r_0=\rho$. Theorem: If $\|H_0\|<1$, then for any λ with $|\lambda|<r_0$ the equation $x=\sum\lambda^kH_kx+y$ has a unique solution which is a limit of the sequence x_n, $x_n=\sum\lambda^kH_kx_{n-1}+y$, x_0 arbitrary. The theorem is applied to the following equations:

$$(1)\qquad u(x)=\int_T G(x, y; \lambda)u(y)dy+F(x),$$

where $G=\sum_0^\infty\lambda^kH_k(x, y)$ and T is a measurable region in n-dimensional Euclidean space in which u and f are square-summable;

$$(2)\qquad x=H_0x+\lambda\sum(\lambda-\lambda_k)^{-1}H_kx+y,$$

where x and y are elements of a Banach space and $\lambda_1, \lambda_2, \cdots$ is a sequence of real numbers, $|\lambda_k|\to\infty$;

$$(3)\qquad u(x)=\int_T[H_0(x, y)+\lambda K(x, y; \lambda)]u(y)dy+f(x),$$

where $K=\sum(\lambda-\lambda_k)^{-1}H_k(x, y)$, the λ_k as above. In the applications various conditions must, of course, be imposed upon the functions or transformations involved in order to make the theorem applicable.

J. V. Wehausen.

Schönberg, Mario. Sur la méthode d'itération de Wiarda et Bückner pour la résolution de l'équation de Fredholm. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 1141–1156 (1951).

Schönberg, Mario. Sur la méthode d'itération de Wiarda et Bückner pour la résolution de l'équation de Fredholm. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 154–167 (1952).

Bückner [Duke Math. J. **15**, 197–206 (1948); Math. Nachr. **2**, 304–313 (1949); these Rev. **9**, 624; **10**, 727] has discussed two methods for obtaining solutions to Fredholm integral equations by iteration. In the first of the papers under review Schönberg shows how these two iterative methods may be extended to apply to equations of the form (1) $y=f+\lambda Ky$, where y and f belong to a Banach space and K is a linear transformation defined over all the space; in doing this some slight extension of the integral equation results is obtained.

In the second paper he notes that both the iteration schemes studied by Bückner may be subsumed under one more general scheme as follows. Let $K=G+H$, where $(I-\lambda H)^{-1}$ exists and may be expanded as $\sum_0^\infty\lambda^nH^n$. Then (1) may be transformed to

$$(2)\qquad y=(I-\lambda H)^{-1}f+\lambda(I-\lambda H)^{-1}Gy=Rf+Qy,$$

where the case of chief interest is $R=R(\lambda K)$ and $Q=Q(\lambda K)$ explicitly. Conditions for the existence of a solution and, in particular, for one of the form $y=\sum_0^\infty[Q(\lambda K)]^nR(\lambda K)f$ are given. Bückner's two schemes correspond to

$$Q=\theta I+(1-\theta)\lambda K$$

and to $Q=\prod_{i=1}^{p}[\theta_iI+(1-\theta_i)\lambda K]$, respectively.

J. V. Wehausen (Providence, R. I.).

Citlanadze, È. S. Proof of the critical point principle for a conditional extremum in a space of type B. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 7–10 (1947). (Russian)

Let $\varphi_i(a)$ $(i=1, 2, \cdots, n)$ and $f(a)$ be functionals defined in a Banach space and having first and second order Fréchet differentials. Let N be the set $\varphi_i(a)=0$ $(i=1, 2, \cdots, n)$. The set of h for which $d\varphi_i(a; h)=0$ $(i=1, \cdots, n)$ is denoted by T_a. A point at which T_a is of deficiency n is called an ordinary point; an ordinary point at which $df(a; h)=0$ for all $h\epsilon T_a$ is called an extreme point. Let $[M]$ denote a "topological class" of sets on N (evidently a class of sets homotopically equivalent) and let c be $\inf_{[M]}\max_M f$, and M_0 a set on which this infimum is attained. Suppose M_0 compact. Then it is shown that the intersection of M with the set $\{f(x)=c\}$ contains at least one critical point of f.

J. L. B. Cooper (Cardiff).

Citlanadze, È. S. On a class of nonlinear operators in the space $l_p(p>1)$. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 533–537 (1948). (Russian)

This article contains results which are the same as those proved for L_p spaces in a later article already reviewed [Doklady Akad. Nauk SSSR **71**, 441–444 (1950); these Rev. **11**, 670] with natural modifications for the case of l_p spaces.

J. L. B. Cooper (Cardiff).

MacColl, L. A. Elementary L_1-spaces. Trans. New York Acad. Sci. (2) **14**, 35–39 (1951).

This paper is concerned with some elementary geometrical properties of an elementary Finsler space, namely, two- and three-dimensional spaces having diamond-shaped unit spheres.

R. S. Phillips (Los Angeles, Calif.).

Cameron, R. H., Lindgren, B. W., and Martin, W. T. Linearization of certain nonlinear functional equations. Proc. Amer. Math. Soc. **3**, 138–143 (1952).

Let C be the space of all real-valued continuous functions $x(t)$ on $[0, 1]$ such that $x(0)=0$, and let $Tx(t)=x(t)+\Lambda(x|t)$ be a (possibly nonlinear) one-one transformation on some Wiener-measurable subset $\Gamma\subseteq C$ in C. It is shown that under suitable conditions on $\Lambda(x|t)$, the equation $y(t)=Tx(t)$ is inverted in the form

$$x(t)=\underset{n\to\infty}{\text{l.i.m.}}\sum_{m_1,\cdots,m_n=0}^{n}A^{(n)}_{m_1\cdots m_n}(t)\prod_{k=1}^{n}\left\{\int_0^1 a_k(s)dy(s)\right\}$$

for each fixed t on $[0, 1]$, l.i.m. being taken in the sense of $L_2(TT)$. Here $\{a_k(s)\}$ is any system of functions closed in $L_2[0, 1]$ and $A^{(n)}_{m_1\cdots m_n}(t)$ is defined as a solution of the system of linear algebraic equations involving $a_k(s)$, $x(s)$, and $Tx(s)$. Of course, the integral $\int_0^1a_k(s)dy(s)$ is to be taken in Paley-Wiener-Zygmund sense. [Cf. another inversion (for the case $\Gamma=C$) of Cameron and Martin, Ann. of Math. **51**, 629–642 (1950); these Rev. **11**, 728.]

K. Yosida (Nagoya).

Berezanskiĭ, Yu. M. Hypercomplex systems with a discrete basis. Doklady Akad. Nauk SSSR (N.S.) **81**, 329–332 (1951). (Russian)

By means of a cubic matrix C_{jkl} with discrete indices and a sequence $\mu_j\geqq0$ such that $\sum_l C_{jkl}C_{lrm}=\sum_l C_{krl}C_{jlm}$, $C_{jkl}=C_{kjl}$,

$\sum|C_{jkl}|\mu_l \leqq \mu_j\mu_k$, the author builds up a hypercomplex system $x=\{x_j\}$, $\|x\|=\sum_j x_j\mu_j$, $(x\cdot y)_l=\sum_{j,k}x_jy_kC_{jkl}$. The author investigates its maximal ideals which are in a one-to-one correspondence with characters $\{t_j\}$ (which are sequences satisfying $\sum_l C_{jkl}t_l\mu_l = t_j\mu_jt_k\mu_k$), proves that the system is isomorphic with a ring of continuous functions and finally applies the theory to find criteria of equivalence between spaces $\Lambda(p(t),\mu)$ for different $p(t)$'s and μ's. The complete space $\Lambda(p(t),\mu)$ is constructed in the following way. $P_j(t)$ are polynomials orthonormal with respect to a given set Q and a weight function $p(t)$, $\mu_j=\max_Q P_j(t)$ and an element of the space is $x(t)=\sum x_jP_j(t)$ where $\sum_{j=0}^\infty |x_j|\mu_j=\|x\|$.

František Wolf (Berkeley, Calif.).

Berezanskiĭ, Yu. M. On the theory of B. M. Levitan's almost periodic sequences. Doklady Akad. Nauk SSSR (N.S.) 81, 493–496 (1951). (Russian)

The author deals with almost periodic sequences generalized by B. M. Levitan [Mat. Sbornik 16(58), 259–280; 17(59), 9–44, 163–192 (1945); same Doklady 58, 977–980, 1593–1596 (1947); Uspehi Matem. Nauk 4, no. 1(29), 3–112 (1949); these Rev. 7, 254; 8, 157; 9, 347; 11, 116]. If

$$(D_kf)_l=\frac{1}{\mu_k\mu_l}\sum_j f_j\mu_jC_{jkl}$$

(where C_{jkl} denotes a cubic matrix satisfying the conditions stated in the preceding review) and M denotes an additive, positive functional determined for all bounded sequences such that

$$\underset{l}{M}[|(D_k\xi)_l|]\leqq \underset{l}{M}[|\xi_l|],\ \underset{l}{M}[\xi_l]=\lim \xi_l$$

whenever the limit exists, then the author proves that for any two almost periodic sequences f, g,

$$\underset{l}{M}[(D_kf)_lg_k]=\sum_t\frac{(f,t)(g,t)}{(t,t)}t_l$$

for almost all l. Here the sum with respect to t refers to all the characters [cf. preceding review]. The inner product is defined as $(f,g)=M[f_l\bar{g}_l]$. The concept "for almost all l" is generated in an obvious way by the functional M. Other results of this paper concern the possibility of approximating an almost periodic sequence by linear combinations of characters. *František Wolf* (Berkeley, Calif.).

Edwards, R. E. The translates and affine transforms of some special functions. J. London Math. Soc. 27, 160–175 (1952).

Let E be the space of all continuous complex functions $f(x)$ on $(-\infty,\infty)$ such that $f(x)\to 0$ as $|x|\to\infty$ with norm $\|f\|=\sup|f(x)|$. The author wishes to determine when certain subsets of E are fundamental. The first theorem reads in part: Let $[f_\omega|\omega\in\Omega]$ be any indexed set of functions belonging to E and having the representation $f_\omega(x)=\int\exp(ix\lambda)F(\lambda)d\lambda$. Let Z_ω, defined modulo null sets, be the set of zeros of $F_\omega(\lambda)$. Suppose given for each ω a set Q_ω of real numbers (in this case $Q_\omega=(-\infty,\infty)$). Finally assume that for each ω, $F_\omega(\lambda)$ is summable. Then the functions $f_\omega(x+q)$ $(\omega\in\Omega,\ q\in Q)$ are fundamental in E provided that the intersection of the sets Z_ω has measure zero. If Ω is uncountable it seems to the reviewer that Z_ω should be redefined as the points of density one for the zero set of $F_\omega(\lambda)$ if the theorem is to remain valid. For functions $f_\omega(x)$ such that $\int\exp(a|\lambda|)|F_\omega(\lambda)|d\lambda<\infty$ (for some $a>0$), the set Q_ω can be considerably diminished. This result is then applied to determine when the affine transforms $[f(px+q)|p\in\Omega,\ q\in Q]$ are fundamental in E. As it stands the statement of this theorem is incomplete; the set Ω must be further qualified. A second part of the paper deals with this same question relative to the locally convex space F of all continuous complex functions on $(-\infty,\infty)$ with the topology of convergence uniform on every compact subset of $(-\infty,\infty)$. *R. S. Phillips* (Los Angeles, Calif.).

Edwards, R. E. Note on the mean-independence of translates of functions. J. London Math. Soc. 27, 249–253 (1952).

Let G with elements $x, y, \cdots$ be a locally compact abelian group and let $f(x)\in L^2(G)$. A subset K of $L^2(G)$ is said to be mean-invariant if whenever $f_1, f_2, \cdots, f_n$ belong to K and $\alpha_1,\alpha_2,\cdots,\alpha_n$ are complex numbers such that $\sum^n|\alpha_i|\leqq 1$, then $\sum_1^n\alpha_if_i\in K$. For an arbitrary set A, the mean-invariant envelope is the smallest mean-invariant set containing A. The translates of a function $f(x)$ are said to be mean-independent if for any neighborhood N of the identity in G, $f(x)$ is not in the mean-invariant envelope of the translates $f(xa)$ with a non-$\in N$. The author shows that if the character group G^* is connected, then every non-trivial function in $L^2(G)$ has its translates mean-independent. In general if $f(x)$ fails to have its translates mean-independent in $L^2(G)$, then $f(x)=p(x)(x,a^*)$, where (x,a^*) is a character of G and $p(x)\in L^2(G)$ is periodic, having as period a non-trivial closed subgroup of G. *R. S. Phillips* (Los Angeles, Calif.).

Wintner, Aurel. On the logarithms of bounded matrices. Amer. J. Math. 74, 360–364 (1952).

The paper deals with infinite bounded [hereafter, bded] matrices of complex numbers. Earlier results of the author [Math. Z. 30, 228–282 (1929)] are recalled: sp A (spectrum of bded A) is not vacuous, is closed and bounded, and is contained in the closure of the set of values of the form $A(x,y)$ under the conditions $y=\bar{x}$, $|x|=(\sum_1^\infty|x_n|^2)^{\frac12}=1$. The question treated is that of the existence of a bded matrix function $f(A)$ for a given A and f; and the particular function $f(z)=\log z$ is considered. The following results are established. (i) Let A be bounded and let the components in the complex plane of the open set $(\text{sp}\,A)^*=$complement sp A be $A^0, A^1, \cdots$, with A^0 as the (unique) unbounded component. To each A^n corresponds a bded non-singular matrix C_n that commutes with A, and for each $\lambda\in A^n$ there is a bded A_λ also commuting with A and such that $\lambda I-A=C_n\exp A_\lambda$. (ii) For each A^n the matrix $\lambda I-A$ has either a bded logarithm for every $\lambda\in A^n$ or for no such λ. (A bded C is called a logarithm of bded A if $A=e^C$.) If $|Z|<1$ then $I+Z$ has a bded logarithm (the series $\sum_1^\infty m^{-1}(-Z)^{m+1}$ being one such). (iii) $\lambda\in A^0$ implies $\lambda I-A$ has a bded logarithm. (iv) If A is completely continuous, then $\lambda I-A$ has a bded, completely continuous logarithm for all λ not in sp A.

Assertion (i) is made to depend on the following lemmas. (I) If A, B commute and are bded, A non-singular and $|B|<|A^{-1}|^{-1}$, then there exists a bded C which commutes with A and B and such that $A+B=Ae^C$. (II) B completely continuous implies that C in (I) can be taken completely continuous. (III) Let A be non-singular and let there exist a 1-parameter family of non-singular mutually commutative matrices $A(t)$, $0\leqq t\leqq 1$ with $A(0)=I$, $A(1)=A$. Suppose $0<\inf|A^{-1}(t)|^{-1}$ $(0\leqq t\leqq 1)$, and that $A(t)$ depends continuously on t (in the sense that

$$|u-v|<\delta_\epsilon \text{ implies } |A(u)-A(v)|<\epsilon).$$

Then there exists a bded C such that $A=e^C$. A final remark points out that the above considerations establish the existence of a logarithm of a finite matrix A (if and only if A is non-singular) without the use of either the Cayley-Hamilton theorem or the reduction to Jordan normal form.
I. M. Sheffer (State College, Pa.).

Krasnosel'skiĭ, M. A. Criteria of continuity of some non-linear operators. Ukrain. Mat. Žurnal 2, no. 3, 70–86 (1950). (Russian)

Let $f(x, u)$ be defined for u real and $x\epsilon G$, a bounded region of Euclidean r-space, and assume f is continuous for x fixed and measurable for u fixed. Let $u(x)\epsilon L_2$ and consider the operator $\mathbf{f}u(x)=f[x, u(x)]$ as a function on L_2 to L_p $(p\geqq 1)$. Theorem 1. This function is continuous if and only if for every sequence $(u_n(x))$ converging to 0 in L_2 the integrals of the functions $|f(x, u_n(x)|^p$ are absolutely continuous uniformly with respect to n. Theorems 2, 3. If the function $\mathbf{f}$ on L_2 to L_p is continuous at one point of L_2 then it is continuous and bounded in every sphere of L_2. Similar theorems are formulated for the corresponding functions on $L_{2,m}=L_2\times L_2\times\cdots\times L_2$ to $L_{p,m}$. Theorem 4. Let $f(x, u)$ satisfy the condition of Theorem 1 and assume $K(x, y)\epsilon L_q(G\times G)$, $p^{-1}+q^{-1}=1$, $1<p\leqq 2$. Then the operator $\mathbf{F}u(x)=\int_G K(x, y)f[y, u(y)]dy$ is defined on L_p to L_2 and is completely continuous. There is also a theorem similar to Theorem 1 on the operator $\mathbf{K}u(x)=\int_G K(x, y, u(y))dy$.
M. Golomb (Lafayette, Ind.).

Krasnosel'skiĭ, M. A. On self-adjoint extensions of Hermitian operators. Ukrain. Mat. Žurnal 1, no. 1, 21–38 (1949). (Russian)

This paper precedes a paper which has already been reviewed [same Žurnal 2, no. 2, 74–83 (1950); these Rev. 13, 47] and makes use of the methods of another by Kreĭn and the author [Uspehi Matem. Nauk 2, no. 3(19), 60–106 (1947); these Rev. 10, 198] to study extensions of operators whose domains are not dense in a Hilbert space $\mathfrak{H}$. For definitions, refer to these reviews, particularly the first. It is shown that the λ-deficiency and λ-semideficiency of an operator are constant inside a connected domain of regular points, and so constant in either half-plane for a hermitian operator. The isometric operators U_λ, the Cayley transform of A, and V_A [loc. cit.] are defined. An allowable extension of U_λ is an isometric extension for which $\tilde{U}_\lambda f=f$ has only the null solution. Hermitian extensions of a hermitian A correspond to allowable extensions of U_λ; these are the extensions which coincide with V_A for no element of $P_{\mathfrak{N}_\lambda}\mathfrak{B}$ save 0. For A to have a selfadjoint extension it is necessary and sufficient that its deficiencies be equal; a general selfadjoint extension A is then of the form

$$A(g+\varphi-U\varphi)=Ag+\bar{\lambda}\varphi-\lambda U\varphi,$$

with $g\epsilon\mathfrak{D}(A)$, $\varphi\epsilon\mathfrak{N}_\lambda$ and U isometric and such that $\mathfrak{D}(U)=\mathfrak{N}_\lambda$, $\mathfrak{R}(U)=\mathfrak{N}_{\bar\lambda}$, $Uf=Vf$ for no element of $P_{\mathfrak{N}_\lambda}\mathfrak{B}$. The existence of a maximal extension for any hermitian operator is also proved; and it is shown that a selfadjoint extension exists operating in a direct sum space $\mathfrak{H}\oplus\mathfrak{H}'$ with the dimension of $\mathfrak{H}'$ at least as great as the deficiencies of A.
J. L. B. Cooper (Cardiff).

Daleckiĭ, Yu. L., and Kreĭn, S. G. On differential equations in Hilbert space. Ukrain. Mat. Žurnal 2, no. 4, 71–91 (1950). (Russian)

The equations considered are of the forms:

(a) $$d\Phi/dt=H(t)\Phi(t),$$

with Φ a variable operator in a Hilbert space, with given initial values; (b) $dq/dt=H(t)q+p(t)$ (in these H is a variable bounded operator, q, p vectors, and in case (b) upper estimates for $\|q\|$ are found when H and p are slowly varying); (c) $Adq/dt=iBq+pe^{i\theta}$, where $A=\sum\epsilon^k A_k(\epsilon t)$, $B=\sum\epsilon^k B_k(\epsilon t)$, $p=\sum\epsilon^k p_k(\epsilon, t)$, $d\theta/dt=k(\epsilon t)$. In case (c) a series expansion asymptotic to the solution is studied; the character of the solution depends on whether $k(t)$ lies in the spectrum of $A^{-1}B$. The operators A_0, B_0 are hermitian, A_0 positive bounded with bounded inverse, and other conditions are laid down. Second order equations of a type similar to (c) are studied by reducing them to the case (c).
J. L. B. Cooper (Cardiff).

Szőkefalvi-Nagy, Béla. The perturbation theory of proper-value problems. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 288–293 (1951). (Hungarian)

An expository lecture on the author's approach to perturbation theory as given, for instance, in Comment. Math. Helv. 19, 347–366 (1947); these Rev. 8, 589.
P. R. Halmos (Chicago, Ill.).

Arf, C. On the methods of Rayleigh-Ritz-Weinstein. Proc. Amer. Math. Soc. 3, 223–232 (1952).

Let Q be a continuous symmetric operator in a Hilbert space H, and R_z its resolvent. A point ζ in the spectrum of Q is considered, subject to a number of requirements which appear, to the reviewer, to be equivalent to the requirement that it be an isolated point of the spectrum. Let P and $\bar{P}$ be complementary projectors. The null manifolds of $Q-\zeta I$ and of $P(Q-\zeta I)P$ are studied, and a number of results are proved concerning the mappings of these on related manifolds, particularly by the operator $\bar{P}R_z\bar{P}$ for z near ζ. If $\bar{P}H$ is finite-dimensional Aronszajn's theorem [Proc. Nat. Acad. Sci. U. S. A. 34, 474–480 (1948); these Rev. 10, 382] concerning the Weinstein determinant $|\bar{P}R_z\bar{P}|$ follows as far as it concerns the behaviour of that function for z near ζ. Further, it is shown that if ρ_z is the resolvent of Q in PH, $\bar{P}R_z\bar{P}$ and $\bar{P}$ $(Q-Iz-Q\rho_z PQ)$ are inverses in $\bar{P}H$: this contains a theorem of Aronszajn on the reciprocal of Weinstein's determinant.
J. L. B. Cooper (Cardiff).

Henriksen, Melvin. On the ideal structure of the ring of entire functions. Pacific J. Math. 2, 179–184 (1952).

The author gives together with other remarks a proof of the following result. Let R be the ring of all entire functions of one variable over the complex field K. If M is any maximal ideal in R, the quotient field R/M is isomorphic to K. As the author points out, if R and R/M are considered as algebras over K, the nature of R/M is no longer the same for all M. In fact, the vector space dimension of R/M is one if M is "fixed", i.e. consists of all entire functions having a common zero, or, equivalently, is a principal maximal ideal, and ∞ if M is "free", i.e. not fixed. [For an analogous study of R/M for rings R of continuous functions see Hewitt, Trans. Amer. Math. Soc. 64, 45–99 (1948); these Rev. 10, 126.]
L. Nachbin (Rio de Janeiro).

Nakano, Hidegorô. Modulared sequence spaces. Proc. Japan Acad. 27, 508–512 (1951).

For fixed sequences $p=(p_n)$ and $w=(w_n)$ with all $p_n\geqq 1$ and all $w_n>0$ let $l(p, w)$ denote the set of all sequences $x=(x_n)$ with finite $\|x\|$ where, by definition, $\|x\|=\inf|\eta|$ with $\sum_n w_n|x_n/\eta|^{p_n}\leqq 1$. Then every $l(p, w)$ is a linear normed space which is complete. $l(p, w)$ and $l(p, w')$ are clearly isometric under the mapping $x_n'=(w_n/w_n')^{1/p_n}x_n$. For two

arbitrary spaces $l(p, w)$, $l(p', w')$ if there is a one-to-one mapping of the form $x_n' = c_n x_n$ then the ratios $\|x'\|/\|x\|$ and $\|x\|/\|x'\|$ must be bounded, considering all corresponding pairs x, x'; hence the c_n must be of the form $c_n = (w_n^{1/p_n}/w_n'^{1/p_n'})k_n$ with the k_n bounded and bounded away from zero (and the k_n can be varied arbitrarily within this restriction). Whether such mappings exist depends on the p, p' but not on w, w' and a necessary and sufficient condition is that for some $0<\alpha<1$, $\sum_n \alpha^{p_n p_n'/|p_n - p_n'|} < \infty$. The author also shows that if $p_n \to 1$ as $n \to \infty$ then $l(p, w)$ has a property first proved by Schur for classical l_1 space, namely: Weak and strong convergence are equivalent. The author's proofs depend on results from his theory of modulared semi-ordered linear spaces but this dependence can be avoided by slight changes which would make his proofs actually somewhat simpler. *I. Halperin* (Kingston, Ont.).

Amemiya, Ichiro. On the equi-continuity in semi-ordered linear spaces. Proc. Japan Acad. 27, 275 (1951).

Let R be a continuous, semi-ordered (that is, σ-complete, partially-ordered) linear space. A linear functional a on R is called universally continuous if infimum $a(x_\lambda)=0$ whenever the x_λ are a directed set with the zero element as intersection, written $x_\lambda \downarrow 0$. For such a form

$$\mu(a) = \sup \{ |a(|x|)| ; \text{all } x \text{ in } K \}$$

for some fixed set K of elements in R. The x in K are called universally equi-continuous if $\inf \mu(a_\lambda)=0$ whenever the a_λ are a directed set with $a_\lambda \downarrow 0$; the x in K are called equi-continuous if this holds whenever the a_λ are also a decreasing sequence.

H. Nakano [Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950, p. 109; these Rev. 12, 420] showed that these concepts of equi-continuity are equivalent for countable sets K (and some restrictions on R). In the present paper this equivalence is proved for all K. Assume equi-continuity. Suppose $a_\lambda \downarrow 0$. Then $\mu(a_\lambda)$ is finite for all λ. A technique due to Nakano [Proc. Imp. Acad. Tokyo 19, 10–11 (1943); these Rev. 7, 249] shows the existence of a sequence λ_n such that $\mu((\wedge a_{\lambda_n}) - a_\lambda(\wedge a_{\lambda_n}))=0$ for all λ which implies $(\wedge a_{\lambda_n} - a_\lambda(\wedge a_{\lambda_n}))x=0$, hence $(\wedge a_{\lambda_n})x=0$, for all x in K. If $[K]$ denotes the projector into the linear manifold spanned by the x in K it will follow that $\wedge(a_{\lambda_n}[K])=0$. The postulated equi-continuity shows therefore that $\inf \mu(a_{\lambda_n}[K])=0$, hence $\inf \mu(a_{\lambda_n})=0$, a fortiori $\inf \mu(a_\lambda)=0$.
I. Halperin (Kingston, Ont.).

Calculus of Variations

De Giorgi, Ennio. Ricerca dell'estremo di un cosiddetto funzionale quadratico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 256–260 (1952).

This paper contains a few remarks on quadratic variation problems. *L. M. Graves* (Chicago, Ill.).

Stampacchia, Guido. Sopra una classe di funzioni in due variabili. Applicazioni agli integrali doppi del calcolo delle variazioni. Giorn. Mat. Battaglini (4) 3(79), 169–208 (1950).

The first part of this paper contains a detailed exposition of closure, convergence, and compactness properties of a class $\mathfrak{a}$ of functions of two variables originally introduced by Nikodym [Fund. Math. 21, 129–150 (1933)], who called them "functions of Beppo Levi." They are closely related to the classes $\mathfrak{P}$ introduced by Morrey [Duke Math. J. 6, 187–215 (1940); these Rev. 1, 209] and Calkin [ibid. 6, 170–186 (1940); these Rev. 1, 208]. A function $u(x, y)$ defined on a region C is said to be in the class $\mathfrak{a}$ if it is absolutely continuous on the interior along almost all lines parallel to the x and y axes, and if its partial derivatives are summable on C. The properties mentioned in the first sentence are studied both on the interior of C and on the closure $\bar{C}$. He gives approximation (by Stieltjes polynomials) theorems for the interior and prolongation theorems from the interior to the closure, the latter especially for a boundary having the property that almost every line parallel to the x or y axis intersects it in finitely many points. Too many definitions would be required to state detailed results. One result is a generalization of a theorem of Scorza-Dragoni [Rend. Sem. Mat. Univ. Padova 17, 102–106 (1948); these Rev. 10, 438]. The second part applies these results to the problem of assuring the existence of the absolute minimum for a double integral problem given by $I(u) = \iint_C f(x, y, u, p, q)\,dxdy$, and succeeds in generalizing some theorems due to Morrey [Theorems 5.1, 5.2, and 5.3 of chap. III of Univ. California Publ. Math. (N.S.) 1, 1–130 (1943); these Rev. 6, 180].
J. M. Danskin (Santa Monica, Calif.).

Magenes, Enrico. Sul minimo relativo degli integrali di Fubini-Tonelli. Giorn. Mat. Battaglini (4) 3(79), 144–168 (1950).

The author develops necessary conditions and also sufficient conditions for a weak relative minimum of the integrals

$$I(y_1, y_2) = \int_c^d \int_a^b f(x, z, y_1(x), y_2(z), y_1'(x), y_2'(z))\,dx\,dz,$$

$$I(y) = \int_a^b \int_a^b f(x, z, y(x), y(z), y'(x), y'(z))\,dx\,dz,$$

in terms of boundary value problems associated with the second variation. For $I(y_1, y_2)$ the boundary value problem is associated with a system of integro-differential equations which is a special case of those considered by Reid [Amer. J. Math. 60, 257–292 (1938)]. For the problem of minimizing $I(y)$, the system of two equations is replaced by a single integro-differential equation. The author briefly compares his boundary value problem for $I(y)$ with that which appeared as a special case in Goldstine's thesis [Contributions to the calculus of variations, 1933–1937, Univ. Chicago Press, 1937, pp. 316–357]. The author also notes that if conjugate points for the problem associated with $I(y)$ are defined in the manner customary in the calculus of variations, in terms of solutions of the accessory integro-differential equation, then a minimizing arc may contain pairs of conjugate points. On the other hand the second variation of $I(y)$ may become negative even if there exists a nonvanishing solution of the accessory integrodifferential equation. *L. M. Graves* (Chicago, Ill.).

Fet, A. I. Variational problems on closed manifolds. Mat. Sbornik N.S. 30(72), 271–316 (1952). (Russian)

Given a positive definite regular variational problem for curves on a closed four times continuously differentiable manifold R, the author establishes the existence on R of a closed extremal and of at least two distinct extremals with given distinct endpoints. He shows also that if the fundamental group of R is finite, then every homotopy class of curves on R with fixed distinct endpoints, contains either two extremals of different lengths, or else continuum-many shortest extremals. The interest of many of the further

results and methods of the paper is somewhat restricted, partly because there is no reference to work outside Russia subsequent to 1936 except translated standard books, and partly because the author assumes all his manifolds and homeomorphisms to be four times continuously differentiable (in a footnote on the second page, he states that he sees no way of dispensing with this assumption).

L. C. Young (Madison, Wis.).

Theory of Probability

Maritz, J. S. Note on a certain family of discrete distributions. Biometrika 39, 196–198 (1952).

Various properties of the distribution of a sum of dependent Poisson random variables. *J. L. Hodges, Jr.*

Arfwedson, G. A probability distribution connected with Stirling's second class numbers. Skand. Aktuarietidskr. 34, 121–132 (1951).

An urn contains N numbered balls. Let $F(N, n, v)$ be the probability that a sample of n balls drawn with replacement from the urn contains v different balls. Writing $F(N, n, v) = \binom{N}{v}\cdot f(n, v)\cdot N^{-n}$ and $f(n, v) = v!g(n, v)$, the numbers $g(n, v)$ are found to be Stirling's second class numbers, i.e. $g(v, n)$ is the coefficient of x^{-n} in $[(x-1)(x-2)\cdots(x-n)]^{-1}$. The author computes the generating function and factorial moments of $F(N, n, v)$ and gives an asymptotic expression for $f(n, v)$ in the case when $v = k\cdot n$ (k constant). Letting $n = cN$ where c is a constant, it is shown that $F(N, cN, v)$ has the limiting normal distribution with zero mean and unit variance. *G. Kallianpur* (Berkeley, Calif.).

Rufener, E. Über eine spezielle Klasse von Frequenzfunktionen. Mitt. Verein. Schweiz. Versich.-Math. 52, 97–120 (1952).

Consider n independent random variables t_i, uniformly distributed for $-a_i < t_i < a_i$, the a_i not being necessarily equal. The author makes a systematic study of the distribution of the sum of the t_i, using mainly the methods of Fourier transforms and geometry. Explicit formulas are derived for the densities. With certain mild restrictions on the a_i the central limit theorem holds as n increases indefinitely. The special case where all $a_i = 1$ is discussed at length. This particular case, as the author points out, was worked out at least as long ago as 1904. For it the reader can refer to H. Cramér, Mathematical methods of statistics, Princeton Univ. Press, 1946, section 19.1, pp. 244–246 [these Rev. 8, 39]. *S. W. Nash* (Vancouver, B. C.).

Solomonoff, R. An exact method for the computation of the connectivity of random nets. Bull. Math. Biophys. 14, 153–157 (1952).

The author defines a matrix whose iterations can be used for computing the weak connectivity of a random net (see the review of Landau's paper below), and also the "strong connectivity", which is the probability of reaching the maximum possible number of points in the net. Previously only approximate formulas had been given.

A. S. Householder (Oak Ridge, Tenn.).

Landau, H. G. On some problems of random nets. Bull. Math. Biophys. 14, 203–212 (1952).

Solomonoff and Rapoport [same Bull. 13, 107–117 (1951); these Rev. 12, 843] define a "random net" as a set of n points, each communicating with a points chosen at random, and the "weak connectivity" γ as the expected fraction of points reached by starting at random and tracing lines of communication. Landau considers an equivalent urn problem with white and black balls, the player paying a ticket to draw a ball, returning always a black ball but receiving a new tickets whenever a white ball is drawn. A difference equation is set up and solved for $\gamma = 1 - E/n$ where n is the total number of balls in the urn, all balls are initially white, the player starts with one ticket, and E is the expected number of white balls when the player has no more tickets.

A. S. Householder (Oak Ridge, Tenn.).

Shimbel, Alfonso. Communication in a hierarchical network. Bull. Math. Biophys. 14, 141–151 (1952).

An analogy is drawn between a hierarchical communication network and a pyramidal election system. An equation is derived relating the probability that a given person will vote for an issue to the probability that the elected officer will also vote for that issue. The equation contains the size of the population, the required majority ratio, and the number of stages in the election system as parameters. (From the author's abstract.) *A. S. Householder.*

Harris, Lee B. On a limiting case for the distribution of exceedances, with an application to life-testing. Ann. Math. Statistics 23, 295–298 (1952).

If a sample of size n is drawn from a population having a continuous (cumulative) distribution function, the author shows in this note that the probability is approximately $1-(1-k)^n$ that $100k\%$ of the elements of a large sample of size N will exceed the largest element in the first sample. He graphs this probability function for 15 values of n ranging from 2 to 1000. *S. S. Wilks* (Princeton, N. J.).

Székely, Gábor. A probabilistic discussion of crushing stones. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 245–248 (1951). (Hungarian)

A. N. Kolmogorov has shown that the distribution of particle sizes in a crushing process is log-normal [C. R. (Doklady) Acad. Sci. URSS 31, 99–101 (1941); these Rev. 3, 4]. The author assumes that the crushing process breaks in each step a randomly chosen particle into eight equal parts and shows that the distribution of particle sizes is asymptotically log-normal. The author mentions that his proof is a modification of a proof by Rényi [Epitöanyag 2, 177–183 (1950)] which was not accessible to this reviewer. *E. Lukacs.*

Takács, Lajos. Discussion of phenomena of occurrence and coincidence in case the distribution of the duration of happenings is arbitrary. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 371–386 (1951). (Hungarian)

The paper deals with the following counter problem. An event (particle entering the counter) produces a "happening" (counter being locked). The following assumptions are made: The events form a Poisson process, an event produces a happening only if it occurs at a time $t > 0$ when no happening is in progress. The happening starts at the instant the event occurs, the duration of the happening (time locked) is a random variable whose distribution is independent of the time of occurrence. This model differs from Feller's type II counter [Courant Anniversary Volume, pp. 105–115, Interscience, New York, 1948; these Rev. 9, 294] in two respects: (1) The time locked is not constant; (2) an event occurring at a moment when the counter is locked does not prolong the inoperative period. The author determines the

expected number of happenings during the interval $(0, t)$ and the expected duration of the happenings starting in the time interval $(0, t)$. Finally he considers a system of such counters and makes a few remarks on coincidences.

E. Lukacs (Washington, D. C.).

Kendall, David G. Some problems in the theory of queues. J. Roy. Statist. Soc. Ser. B. 13, 151–173; discussion: 173–185 (1951).

This expository article contains an extensive discussion of the properties of waiting lines before a single gate or counter where service is given. The restriction to a single gate seems to be necessary, since even in this simplest case the proofs of the existence of limits in their dependence on the loading of the system (measured by the ratio of the average rate of arrivals to that of departures) are lengthy. Some remarks are made on the many-gate problem, essentially the problem of telephone traffic treated by the pioneers of this subject, though not of course in full generality. More specifically, the system considered has arrivals falling at random (Poisson variables), service in order of arrival, and general service-time distribution. The restriction to Poisson arrivals has since been removed by one of the discussors, D. V. Lindley [Proc. Cambridge Philos. Soc. 48, 277–289 (1952); these Rev. 13, 759]. The expected values of the length of waiting line and the waiting time are first determined and then the Laplace transform of the equilibrium distribution of waiting times, a result due to Pollaczek [Math. Z. 32, 64–100, 729–750 (1930)], and by a different method to Khintchine [Mat. Sbornik 39, no. 4, 73–84 (1932); 40, 119–123 (1933)]. A long discussion of the approach to equilibrium as it depends on the load then appears; the approach and terminology of Feller's work on recurrent events and Markov chains [Trans. Amer. Math. Soc. 67, 98–119 (1949); An introduction to probability theory and its applications, vol. 1, Wiley, New York, 1950, ch. 13; these Rev. 11, 255; 12, 424] are used. Finally, the work of Borel [C. R. Acad. Sci. Paris 214, 452–456 (1942); these Rev. 4, 248] on the length of busy periods and of customers served in each, which was done for exponential service time distribution is generalized to general service time distribution through an interesting interpretation by a branching process. An alternative proof is given in the discussion by I. J. Good.

As the discussion emphasizes, the material has numerous applications, to aircraft landing, road traffic, ship docking, railroad station taxi loading, for example. An instructive set of graphs appears in the discussion by F. G. Foster, comparing the behaviour of systems with constant and exponential service time distributions, both by length of waiting line and by waiting time, the data being obtained by statistical experiment with random numbers (in telephone parlance, a "throwdown").

J. Riordan.

Pollaczek, Félix. Fonctions caractéristiques de certaines répartitions définies au moyen de la notion d'ordre. Application à la théorie des attentes. C. R. Acad. Sci. Paris 234, 2334–2336 (1952).

Let $X_1, X_2, \cdots$ be independent random variables with the same distribution function, $S_n = X_1 + X_2 + \cdots + X_n$, $X_{n,m} = \max^{(m)}(S_1, S_2, \cdots, S_n)$, where $\max^{(m)}$ is the mth largest ($\max^{(1)} = \max$), $\max^{(m)+} = \max(\max^{(m)}, 0)$, and

$$\varphi_{n,m}(-q) = E[\exp(-qX^+_{n,m})];$$

the author gives a formula for the generating function

$$F(q, x, y) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} x^{m-1} y^n \varphi_{n,m}(-q),$$

namely

$$F(q, x, y) = \frac{1}{(1-x)(1-y)} \times \exp\left[\frac{q}{2\pi i} \int_{-i\infty-0}^{i\infty-0} \log \frac{1-xy\varphi(-s)}{1-y\varphi(-s)} \frac{ds}{s(q-s)}\right].$$

The corresponding generating function for random variables $X^+_{n,m;c} = -c + \max^{(m)+}(c+S_1, \cdots, c+S_n)$ is related to this in simple fashion. Finally, a generating function for two similar correlated variables is given. These results are applied to the delay distribution in a waiting line before a single gate.

J. Riordan (New York, N. Y.).

Jánossy, Lajos. A generalization of the Laplace transform in probability theory. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 343–350 (1951). (Hungarian)

Let $a(x, y)$ (respectively $A_k(x, y)$) be the one step (respectively k-step) transition probabilities of a stationary Markov process. The function $a(x, y)$ is taken as the kernel of the integral equations (1) $\phi_s(y) = s\int \phi_s(x)a(x, y)dx$ and (2) $\psi_s(x) = s\int \psi_s(y)a(x, y)dy$. It is shown that

$$\phi_s(y) = s^k \int \phi_s(x) A_k(x, y) dx$$

(with a similar equation holding for $\psi_s(x)$). The author expresses then the k-step transition probability in terms of the eigenfunctions of the equations (1) and (2) and uses this expansion to study the process. He also shows that the method reduces to the use of the Laplace transform in case $a(x, y) = a(x-y)$. Much of the material may be found in Hostinsky's book [Méthodes générales du calcul des probabilités, Mémor. Sci. Math., no. 52, Gauthiers-Villars, Paris, 1931, especially pp. 46–47].

E. Lukacs.

Jánossy, Lajos. A statistical problem of cascade processes. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 213–217 (1951). (Hungarian)

This is essentially a presentation of the work published recently by the author [Proc. Phys. Soc. Sect. A. 63, 241–249 (1950)] and by the author jointly with H. Messel [ibid. 63, 1101–1115 (1950)] on the fluctuation of cascades.

E. Lukacs (Washington, D. C.).

Jánossy, Lajos. A stochastic process occurring in the theory of the multiplier tube. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 357–367 (1951). (Hungarian)

The paper deals with the classical branching process. The author studies the process by means of its generating function and derives its asymptotic properties in the usual manner [see Feller, An introduction to probability theory . . . , vol. I, Wiley, New York, 1950, pp. 223–227; these Rev. 12, 424]. He derives asymptotic formulae for the moment generating function (and its derivatives) of the distribution of the number of individuals in the Nth generation. Finally he assumes that the number of individuals in the first generation has a Poisson distribution (with specified parameter) and tabulates values of the above mentioned functions for this particular case.

E. Lukacs.

Rényi, Alfréd, and Turán, Pál. Two proofs of a theorem of L. Jánossy. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 369–370 (1951). (Hungarian)

In the paper reviewed above Jánossy proved the following theorem: "Let $G(z) = \sum_{k=0}^{\infty} p_k z^k$ and assume that $p_0 > 0$,

$p_k \geqq 0$ $(k=1, 2, \cdots)$ and $G(1)=1$, then the equation $G(z)=z$ has at most one root in the interior of the unit circle. This root can only be real and positive." The first author gives an elementary algebraic proof while the proof of the second author uses Rouché's theorem. The theorem is however not new, it may be found in Feller's book [An introduction to probability theory . . . . , Wiley, New York, 1951, p. 226; these Rev. 12, 424] where it occurs (as in Jánossy's paper) in connection with branching process. *E. Lukacs.*

Rényi, Alfréd. On problems connected with the Poisson distribution. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 202–212 (1951). (Hungarian)

This is an expository lecture on the Poisson distribution. Applications of this distribution in telephone engineering are discussed in detail, other applications are briefly mentioned. *E. Lukacs* (Washington, D. C.).

Jánossy, L., Rényi, A., and Aczél, J. On compound Poisson distributions. I. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 315–328 (1951). (Hungarian)

This is the Hungarian version of a paper by the same authors [Acta Math. Acad. Sci. Hungar. 1, 209–224 (1950); these Rev. 13, 663]. *E. Lukacs* (Washington, D. C.).

Rényi, Alfréd. On compound Poisson distributions. II. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 329–341 (1951). (Hungarian)

This is the Hungarian version of a paper by the same author [Acta Math. Acad. Sci. Hungar. 2, 83–98 (1951); these Rev. 13, 663]. *E. Lukacs* (Washington, D. C.).

Blanc-Lapierre, André. Remarques sur un théorème d'interpolation. C. R. Acad. Sci. Paris 234, 1733–1735 (1952).

Let $X(t)$ be a real strictly stationary random function of second order, with null expectation. Assume that the spectral distribution function, determined by the covariance of $X(t)$, varies only in a bounded interval $(-a, +a)$. Assume that

$$X(t)=\sum_{-\infty}^{\infty} X(t_0+n\lambda)\varphi(t-t_0-\lambda n)$$

where the summation is taken in the sense of limit in quadratic mean. If the random variables $X(t_0+n\lambda)$ are orthogonal, then $\lambda=1/2a$ and $\varphi(t)=(2\pi at)^{-1}\sin 2\pi at$. If the random variables $X(t)$ are independent, then, moreover, the random function is normal. *M. Loève* (Berkeley, Calif.).

Ryll-Nardzewski, Cz., et Steinhaus, H. Sur les fonctions indépendantes. IX. Séries des fonctions positives. Studia Math. 12, 102–107 (1951).

Two measurable, real-valued functions $f(\tau)$ and $g(\tau)$, $0\leqq\tau\leqq1$, are called equivalent if they have the same distribution function, i.e., for all real α the sets where $f(\tau)<\alpha$ and $g(\tau)<\alpha$ have the same measure. Two series $\sum_1^\infty f_n(\tau)$ and $\sum_1^\infty g_n(\tau)$ are called equivalent if f_n and g_n are equivalent for every n. The authors prove the following interesting theorem: Consider the series equivalent to $\sum f_n(\tau)$, where $f_n(\tau)\geqq0$; then the measure of the set of convergence is minimum for the series whose terms are non-decreasing functions and maximum for the series whose terms are statistically independent. *M. Kac* (Ithaca, N. Y.).

√★**Fortet, Robert. Random functions from a Poisson process.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 373–385. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let $N_1(t)$ $(t\geqq0)$ be the Poisson process with density $m>0$ and set $N(t)=N_1(t)$, $t\geqq0$ and $N(t)=-N_1(t)$, $t<0$. The present paper is devoted to a study of random functions of the form $X(t)=\int_{-\infty}^{\infty}R(t,\tau)dN(\tau)$. Conditions under which $X(t)$ becomes Gaussian (in an appropriate limit) are found. Special attention is paid to the case $R(t,\tau)=R(t-\tau)$. *M. Kac* (Ithaca, N. Y.).

Fortet, R. On some functionals of Laplacian processes. J. Research Nat. Bur. Standards 48, 32–39 (1952).

Let $N(t)$ be a Poisson process with constant parameter m. With a real function $R(t,\tau)$ the author associates a process $X(t)$ obtained as a stochastic integral,

$$X(t)=m^{-1/2}\int R(t,\tau)dN^*(\tau)$$

with respect to the centered Poisson process $N^*(t)$. In a previous paper [see the preceding review] the author proved that, as $m\rightarrow\infty$, $X(t)$ converges in the sense of distribution to a Laplacian process. In this paper the author proves that, conversely, under certain conditions a Laplacian process can be represented as a limit from a Poisson process and some $R(t,\tau)$ as above. The author proves that for $X(t)$ as above the distribution of a functional of the type $\int_t^{t+\tau}V(u, X(u))du$ under certain conditions tends to a Laplacian distribution as $\tau\rightarrow\infty$. This result is then extended to the case that $X(t)$ is itself a Laplacian process by the above mentioned representation. *J. L. Snell* (Princeton, N. J.).

Rosenblatt, M. The behavior at zero of the characteristic function of a random variable. Proc. Amer. Math. Soc. 3, 498–504 (1952).

Let X be a random variable with distribution function $F(x)$ and characteristic function $\varphi(z)$. The following behaviour of $\varphi(z)$ at $z=0$ is obtained, as corollary of a more general theorem which is derived from the weak law of large numbers for independent and identically distributed random variables. For $0<\alpha<2$, $1-\varphi(z)=o(|z|^\alpha)$ if, and only if, (i) $\int_{|x|>n}x^{1/n}dF(x)=o(1)$, and (ii) $\int_{|x|<n}xdF(x)=o(1)$ if $\alpha=1$, (iii) $\int xdF(x)=0$ if $\alpha>1$. *M. Loève.*

Gnedenko, B. V. On some properties of limiting distributions for normed sums. Ukrain. Mat. Žurnal 1, no. 1, 3–8 (1949). (Russian)

Let $\xi_1, \xi_2, \cdots$ be independent random variables. If for some constants A_n and $B_n>0$ the distribution function (d. f.) of $B_n^{-1}(\xi_1+\cdots+\xi_n)-A_n$ converges to a limit and for some constants b_{nk} the variables $B_n^{-1}\xi_k-b_{nk}$ $(1\leqq k\leqq n)$ converge to zero in probability, then the limit is said to belong to class L. Complete characterization of class L as a subclass of infinitely divisible laws was given by Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937, p. 192]. It includes all the stable laws. Theorem 1. If $F(x)$ belongs to the domain of attraction of a stable law of exponent α, then for all $\delta<\alpha$, $\int|x|^\delta dF(x)<\infty$. This follows from the necessary and sufficient conditions of Doeblin-Gnedenko (see, e.g., the book cited below). Following Khintchine a d. f. is called unimodal if there exists a real a such that $F(x)$ is convex for $x<a$ and concave for $x>a$. Theorem 2. Every d. f. of class L is unimodal. Even in the case of stable laws this was an unsolved problem. The proof

depends on a result of Lapin [1947, in an unavailable dissertation] which states that the convolution of two unimodal d. f.'s is unimodal which in turn depends on Khintchine's characterization of a unimodal d. f. as one for which $F(x)-xF'(x)$ is a d. f. (1938). Now, for a given $F(x)$ of class L, independent, identically distributed random variables ξ_{nk} $(1\leqq k\leqq n)$ are constructed such that the d. f. of each is unimodal and the d. f. of $\xi_{n1}+\cdots+\xi_{nn}-A_n$ converges to $F(x)$. The convergence is established by Gnedenko's general convergence theorem for infinitely divisible laws. Thus Theorem 2 is proved by Lapin's result and the trivial one that the limit of unimodal d. f.'s is unimodal. All these results are given in §32 of Gnedenko and Kolmogorov, Limit distributions for sums of independent random variables [Moscow-Leningrad, 1949; these Rev. 12, 839].

K. L. Chung (Ithaca, N. Y.).

Takahashi, Shigeru. On the asymptotic distribution of the sum of independent random variables. Proc. Japan Acad. 27, 393–400 (1951).

Let $X_1, X_2, \cdots$ be mutually independent random variables with

$$(*)\qquad \lim_{n\to\infty} P\left\{\sum_1^n X_i(\omega)\leqq an^{1/2}\right\}=G(a),$$

where G is the normal distribution function with 0 mean and unit variance. Let $\mathfrak{F}$ be the smallest Borel field of ω-sets with respect to which the X_i's are all measurable. Then if $E\epsilon\mathfrak{F}$ it is proved that

$$\lim_{n\to\infty} P\left\{\sum_1^n X_i(\omega)\leqq an^{1/2},\ \omega\ \epsilon\ E\right\}=P\{E\}G(a).$$

[Note by reviewer. The simplest proof is probably that the class of $\mathfrak{F}$-sets E for which the theorem is true is closed in the usual set-metric, and includes sets determined by inequalities imposed on any finite number of X_i's, hence is $\mathfrak{F}$.] Equation (*) is then generalized (1) by replacing $an^{1/2}$ by $g(\omega)n^{1/2}$, where g is non-negative and $\mathfrak{F}$ measurable, and (2) by summing over $nN(\omega)+O(n^{1/2})$ variables, instead of n, where N is $\mathfrak{F}$ measurable. In both cases the new right side of (*) is obtained explicitly by use of the first cited theorem. For earlier results in this connection, see J. C. Smith [Bull. Amer. Math. Soc. 51, 941–944 (1945); these Rev. 7, 209] and H. Robbins [ibid. 54, 1151–1161 (1948); these Rev. 10, 385].

J. L. Doob (Urbana, Ill.).

Milicer-Grużewska, Halina. On the law of probability and the characteristic function of the standardized sum of equivalent variables. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 42 (1949), 99–143 (1952). (English. Polish summary)

The author continues her earlier investigations [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) 2, 25–33 (1948); these Rev. 11, 118] of limit theorems for correlated equivalent random variables. A number of theorems valid for the distribution functions and for the characteristic functions of standardized sums of equivalent random variables are proven and the approximation to the normal distribution is discussed.

E. Lukacs.

Lévy, Paul. Systèmes markoviens et stationnaires. Cas dénombrable. Ann. Sci. École Norm. Sup. (3) 68, 327–381 (1951).

Let $H(t)$ be a Markov chain or process with stationary transition probabilities and denumerably many states A_h. We call it a chain or a process according as t ranges over all non-negative integers or real numbers. Let $H(t)=k$ if $H(t)$ is in the state A_k. The transition probability is $P_{hk}^{(n)}$ or $P_{hk}(t)$.

Chapter 1 deals with chains. A resumé of the most important results, mostly due to Kolmogorov [Mat. Sbornik 1(43), 607–610 (1936); Bull. Math. Univ. Moscou 1, no. 3, 1–16 (1937)], is given with proofs. A new result states: In a final ergodic group (= recurrent class) the ratio of the frequencies of the two states A_h and A_k tends almost surely to β_{hk}/β_{kh} where β_{hk} is the probability that between two consecutive roots of $H(n)=h$ there is at least one root of $H(n)=k$. This is the "strong" analogue of Doeblin's ratio ergodic theorem [Bull. Soc. Math. France 66, 210–220 (1938)] but neither implies the other. A highly original proof of the main ergodic theorem is given. This depends on the following idea. Let X be the recurrence time of A_0. The $P_{00}^{(n)}$ for all n are determined by the distribution of X. It follows that in order to prove that $P_{00}^{(n)}$ tends to a limit as $n\to\infty$ it is sufficient to choose any simple type of chains such that all possible distributions for X are realized by one chain of this type. One such choice is given by $P_{h0}^{(1)}=\alpha_h$, $P_{h,h+1}^{(1)}=1-\alpha_h$, $h=0, 1, 2, \cdots, 0\leqq\alpha_h<1$. The proof of the theorem is then accomplished by using the result in the finite case. Whilst this proof is not the shortest known the ideas used may be useful elsewhere.

Chapter II and III deal with processes. The treatment is new and self-contained except for several results of Doob [Trans. Amer. Math. Soc. 52, 37–64 (1942); 58, 455–473 (1945); these Rev. 4, 17; 7, 210]. The probability that if $H(0)=h$, $H(\tau)$ should equal h for all $0<\tau<t$ is seen to be $e^{-\lambda_h t}$ with $0\leqq\lambda_h\leqq\infty$. If $\lambda_h=\infty$, A_h is called an instantaneous state. For a fixed sample function $H(t)$ consider on the t-axis the successive open intervals of sojourn in the different states; the complement of the union of these intervals form a closed set $\mathfrak{E}$ which is the set of points of discontinuity of $H(t)$. For processes without instantaneous states (types 1–4 below) $\mathfrak{E}$ has almost surely measure 0. Let T be the time required to run through a succession of states consisting of n_h realizations of A_h (order immaterial). Theorem 1. T is almost surely finite or infinite with its expectation $\sum n_h\mu_h$ $(\mu_h=\lambda_h^{-1})$. Theorem 2. If the last series converges, then T has an absolutely continuous distribution with positive density from 0 to ∞.

Let E_h be the set of t when $H(t)=h$. A sample function $H(t)$ is said to belong to the type 1, 2, 3, 4, 5, 6 according as 1) $\mathfrak{E}$ has no finite limit point; 2) $\mathfrak{E}$ is well-ordered but has at least one finite limit point (thus $H(t-0)$ fails to exist for some t but $H(t+0)$ always exists); 3) $\mathfrak{E}$ is denumerable but not well-ordered (thus $H(t+0)$ fails to exist for some t); 4) $\mathfrak{E}$ is of measure 0 but not denumerable; 5) All E_h are measurable but at least one of them is of positive measure; 6) At least one E_h is not measurable. A process is of type n $(n\leqq6)$ if all $P_{hk}(t)$ are measurable and, for at least one choice of $H(0)$, $H(t)$ is of type n with positive probability but for no choice of $H(0)$ is $H(t)$ of a higher type with positive probability. It is of type 7 if some $P_{hk}(t)$ is not measurable.

Type 2 was discovered by Doob [1945 paper] and types 1 and 2 have been exhaustively studied by him. Examples of types 3 and 4 are given here using the idea of a "fictitious state". Let p_{hk} be the probability that at the moment $H(t)$ quits A_h it enters A_k. Consider a process with $p_{h,h+1}=1$ for $h=0, \pm1, \pm2, \cdots$. If $\sum\mu_h<\infty$ then by Theorem 1 above the time required to run through all the positive-indexed states, starting from A_0, is almost surely finite. At the end of this time let it pass instantly from the fictitious state $+\infty$

to $-\infty$. At this instant neither $H(t-0)$ nor $H(t+0)$ exists. It should be remarked that for type 2 a fictitious state is remindful of Doob's formal construction (loc. cit.) known as "the ghostlike return after an explosion" (Feller). Its adaptation to the other types is new and merits further analysis. There is a general discussion of fictitious states and other examples. Modifying another example somewhat for visual ease one may construct a type 4 process using the well-known Cantor set E (prolonged periodically to infinity) as follows. Randomize the length of each of the open intervals whose union forms the complement of E. These are made the sojourn-time for the non-instantaneous states (we may establish a one-to-one correspondence between the intervals and the states). At each limit point t of E neither $H(t-0)$ nor $H(t+0)$ exists. At the left (right) endpoint of each of the open intervals $H(t-0)$ ($H(t+0)$) does not exist. They correspond to different fictitious states. To insure the return from a fictitious state Theorem 1 above must be used.

From Baire's lemma it follows that there are infinitely many non-instantaneous states and that the set of sojourn-time in them is everywhere dense on the t-axis. A constructive definition of the most general processes of type 4 and 5 is indicated by introducing a new parameter which keeps time as it were on the instantaneous states. In case of type 5 if A_0 is the only state whose E_0 is of positive measure the new parameter τ may be taken to be the measure of $E_0\cap(0, t)$. The process $X(\tau)=t-\tau$ is of independent stationary increments and thus of a known structure. Open problems are mentioned in this connection. Every process of type 6 is equivalent to one with double-indexed indices $A_{k,l}$ such that the first index follows a process of type $\leqq 5$ while the second is such that $P_{l,l'}(t)=P_{l'}$, independently of l and t. An example of type 7 was given by Doob [1942 paper] using a Hamel basis. This can be slightly generalized and mixed with the other types, but it is not known if this exhausts all possibilities.

For types $\leqq 6$ the following theorems are proved. Theorem 3. Each $P_{kk}(t)$ is either identically 0 or positive for all $t>0$. This answers a question raised by the reviewer. The proof is simple enough for the first 4 types; for type 5 some amplification seems to be needed; for type 6 it follows from the characterization mentioned above. Theorem 4. $\lim_{t\to\infty} P_{hk}(t)=P_{hk}$ exists. The elegant proof of theorem 4 is based on the ergodic theorem for chains and the fact that $P_{hk}(t)$ is uniformly continuous for $t\geqq t_0>0$. This last fact follows from results of Doob (1942) but was overlooked by him.

Four Comptes Rendus notes were the forerunners of this paper. [C. R. Acad. Sci. Paris **231**, 467–468, 1208–1210 (1950); **232**, 1400–1402, 1803–1805 (1951); these Rev. **12**, 269, 619, 723, 840.] *K. L. Chung* (Ithaca, N. Y.).

Wasow, Wolfgang. On the mean duration of random walks. J. Research Nat. Bur. Standards **46**, 462–471 (1951).

Using methods closely related to those of Petrovsky [Math. Ann. **109**, 425–444 (1934)] the author treats the problem of finding the mean duration of a random walk in a domain G, assuming fairly general transition probabilities. In the usual limit one is led to the equation $L[V]+1=0$ with boundary value 0 on the boundary of G. Here L is an elliptic operator which is derived from the infinitesimal properties of transition probabilities. In the special case of an ordinary random walk L is the Laplacian. In the latter case estimates are obtained for the mean duration. Such estimates are useful in evaluating the usefulness of sampling methods in solving the Dirichlet problem. *M. Kac.*

Goldstein, S. On diffusion by discontinuous movements, and on the telegraph equation. Quart. J. Mech. Appl. Math. **4**, 129–156 (1951).

The following random walk is discussed: A particle, starting from the origin, moves in steps of length $\Delta=v\tau$, the duration of each step being τ. At each time the particle has probability p of maintaining and the probability $q=1-p$ of reversing the direction of the previous step (at $\tau=0$ both directions are equiprobable). The probability distribution of the position is calculated exactly and various asymptotic expressions are found. In one particular limit the solution can be obtained by solving the "telegraph equation". This is remarkable inasmuch as the equation is hyperbolic and one usually encounters equations of parabolic type.

M. Kac (Ithaca, N. Y.).

Matschinski, Mathias. Sur la probabilité des hypothèses C. R. Acad. Sci. Paris **234**, 1428–1430 (1952).

Matschinski, Matthias. Sur la probabilité de l'hypothèse de périodicité. C. R. Acad. Sci. Paris **235**, 14–17 (1952).

Földes, István. Applications of the calculus of probability in astronomy. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 235–244 (1951). (Hungarian)

The author lists a few astronomical problems where use was made, by various authors, of the calculus of probability, and briefly outlines the procedure which was followed in some of these cases. Schwarzschild's stellar statistics, the statistical treatment of proper motions and radial velocities in the galactic system and the problem of meteor radiants are briefly mentioned. A more detailed description is given of the method by which Jeans and Ambarzumian analyzed the probability of near-encounters in the galactic system. Problems of stellar dynamics, such as those connected with the relaxation time of the galactic system, are summarily examined at the end of this brief survey. *L. Jacchia.*

★La cybernétique. Théorie du signal et de l'information. Réunions d'études et de mises au point tenues sous la présidence de Louis de Broglie. Editions de la Revue d'Optique, Paris, 1951. vi+318 pp. 1600 francs.

Proceedings of a series of lectures held in April and May, 1950. Table of contents: Introduction (pp. 1–7) by J. Loeb; Processus stationnaires et de Markoff et entropie (pp. 9–33) by R. Fortet; Le filtrage et la prédiction des messages selon Norbert Wiener (pp. 35–53) by M. D. Indjoudjian; Le bruit dans les télécommunications (pp. 55–81) by A. Blanc-Lapierre; La transmission de l'information (pp. 83–102) by P. Aigrain; Les signaux analytiques et leurs transformations (pp. 103–114) by J. Oswald; La théorie des communications et la physique (pp. 115–149) by D. Gabor; Mesure de la quantité d'information (pp. 151–160) by J. Ville; L'application des moyens d'analyse de la qualité des transmissions téléphoniques (pp. 161–195) by P. Chavasse; Pertes d'information par dispersion (pp. 197–206) by S. Colombo; La télévision et la transformation des informations (pp. 206–235) by Y. Delbord; Multiplex et théorie des communications (pp. 237–253) by J. Icole; Compression de fréquences (pp. 255–296) by P. Marcou; Codage et information (pp. 297–314) by E. Picault.

Féron, Robert. Convexité et information. C. R. Acad. Sci. Paris **234**, 1840–1841 (1952).

If the amount of information in a distribution function F, with density f, is defined as $\int_{-\infty}^{\infty} H[f(y)]dy$, where H satisfies unspecified regularity conditions, then the information in a distribution with density $cf(cy)$ is monotone decreasing in c if and only if H is concave. More generally, if the information is defined as $\Phi(F)$, and if the functional Φ is called concave if (*) $\Phi[\frac{1}{2}(F_1+F_2)] \geqq \frac{1}{2}[\Phi(F_1)+\Phi(F_2)]$, then for every pair of random variables X, Y the information in the Y distribution is at least the expected value of the information in the Y distribution for given X if and only if Φ is concave. There is equality here if X, Y are independent, and this is the only case of equality if and only if there is equality in (*) only when $F_1=F_2$. *J. L. Doob.*

Middleton, David. On the distribution of energy in noise- and signal-modulated waves. II. Simultaneous amplitude and angle modulation. Quart. Appl. Math. **10**, 35–56 (1952).

Middleton, David. The distribution of energy in randomly modulated waves. Philos. Mag. (7) **42**, 689–707 (1951).

Mathematical Statistics

Nair, K. R. Tables of percentage points of the 'Studentized' extreme deviate from the sample mean. Biometrika **39**, 189–191 (1952).

Denote by $x_{(1)}, \cdots, x_{(n)}$ a random sample of n observations from a normal population with standard deviation σ. Let $x_1, \cdots, x_n$ be the same sample arranged in ascending order of magnitude so that x_r is the rth ranked (or ordered) variate in the sample $\{x_{(i)}\}$. Let an estimate s_v of the unknown σ be available with v degrees of freedom independent of the sample $\{x_{(i)}\}$. The author previously published the lower and upper 5% and 1% points of $(x_n-\bar{X})/s_v$ or $(\bar{X}-x_1)/s_v$ for selected values of n and v [Biometrika **35**, 118–144 (1948); these Rev. **9**, 602]. The present extended tables give the lower 10%, 5%, 2.5%, 1%, 0.5%, and 0.1%, points of the "Studentized" extreme deviate $(x_n-\bar{X})/s_v$ or $(\bar{X}-x_1)/s_v$ for $n=3(1)9$, $v=10, 15, 30, \infty$ to two decimals, and the upper 10%, 5%, 2.5%, 1%, 0.5%, and 0.1% points of the same statistics for $n=3(1)9$, $v=10(1)\ 20, 24, 30, 40, 60, 120$, and ∞ to three significant figures.
L. A. Aroian (Culver City, Calif.).

May, Joyce M. Extended and corrected tables of the upper percentage points of the 'Studentized' range. Biometrika **39**, 192–193 (1952).

Let $x_1, x_2, \cdots, x_n$ be a random sample of n items from a normal population with standard deviation σ arranged in ascending order of magnitude. Let $w=x_n-x_1$, the sample range, and $q=w/s$, where s^2 is a mean square estimate of σ^2, independent of w and based on v degrees of freedom. The upper 5% and 1% points of q are given usually to 3 significant figures and occasionally to 4 for $n=2(1)20$, and $v=1(1)20, 24, 30, 40, 60, 120, \infty$. The results were obtained by numerical quadrature and were checked for $1\leqq v\leqq 4$, $2\leqq n\leqq 8$, against the values found by Pillai by another method [see the following review]. This table supersedes a previous smaller one by Pearson and Hartley [Biometrika **33**, 89–99 (1943); these Rev. **6**, 92], some values of which were in error. *L. A. Aroian* (Culver City, Calif.).

Pillai, K. C. S. On the distribution of 'Studentized' range. Biometrika **39**, 194–195 (1952).

The notation is that of the preceding review. The author determines the distribution of q explicitly in the form of an infinite series. He gives the lower 5% and 1% points of q for $n=2(1)8$, $v=1(1)9$ to 2 decimal places. The upper percentage points are included in the paper reviewed above.
L. A. Aroian (Culver City, Calif.).

Masuyama, Motosaburo. An improved binomial probability paper and its use with tables. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, 15–22 (1 plate) (1951).

R. A. Fisher [Proc. Roy. Soc. Edinburgh **42**, 321–341 (1922)] introduced the "angular transformation" of multinomial variates which is the basis of the binomial probability paper which was designed by Mosteller and Tukey and whose use was described by them [J. Amer. Statist. Assoc. **44**, 174–212 (1949)]. The paper is graduated with a square-root scale on both axes and is used for plotting the point $(\sqrt{n_1}, \sqrt{n_2})$ where n_1 and n_2 are the numbers of the two kinds of observations in a sample from a binomial population. Various significance levels, confidence intervals and other quantities can then be found graphically. The improved paper of M. Masuyama has several auxiliary tables and scales printed in the margin which extend its usefulness, along with some improvements in design. The present article describes its uses. *T. E. Harris.*

Cadwell, J. H. An approximation to the symmetrical incomplete beta function. Biometrika **39**, 204–207 (1952).

Define

$$\Phi(\zeta)=\int_0^\zeta \frac{e^{-\frac{1}{2}t^2}}{(2\pi)^{1/2}}dt, \quad \Psi(\zeta)=\int_0^\zeta \left(1-\frac{t^6}{15}\right)\frac{e^{-\frac{1}{2}t^2}}{(2\pi)^{1/2}}dt,$$

$$x=\tfrac{1}{2}+\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\Phi\left\{\left(\frac{3}{4p-1}\right)^{\frac{1}{2}}\zeta\right\}.$$

The author shows that $I_x(p, p)$ is approximately equal to

$$\tfrac{1}{2}+L(p)\left\{\Phi(\zeta)+\frac{2(p-1)(2p+1)}{(4p-1)^4}\Psi(\zeta)\right\},$$

where $L(p)$ is given in tabular form. For $p=2$, the maximum error of the approximation is 0.00011 and for $p>5$, it will give 5 figure accuracy. Applications are made to the Pearson type II and type VII curves. *L. A. Aroian.*

Cadwell, J. H. The distribution of quantiles of small samples. Biometrika **39**, 207–211 (1952).

Consider the ratio of the standard error of the median to the standard error of the mean for n independent random variables drawn from the same normal population. The author finds improved approximate formulas for this ratio as polynomials in n^{-1}. Numerical examples show very good agreement with exact results due to T. Hojo [Biometrika **23**, 315–360 (1932)]. The author also indicates how to derive the corresponding ratio for any quantile where the population sampled has a continuous density.
S. W. Nash (Vancouver, B. C.).

Rider, Paul R. The distribution of the range in samples from a discrete rectangular population. J. Amer. Statist. Assoc. **46**, 375–378 (1951).

The distribution is derived, and an application of it in testing randomness of entries in a random number table is pointed out. A table of the distribution for population size 10 and sample sizes $2, \cdots, 10$ is also given. The author was

apparently unaware that the distribution, a similar application, and most of the table are given in a paper by E. L. Dodd [Econometrica 10, 249–257 (1942), pp. 254–255; these Rev. 4, 108]. *D. F. Votaw, Jr.*

Pearson, E. S. Comparison of two approximations to the distribution of the range in small samples from normal populations. Biometrika 39, 130–136 (1952).

The author investigates the adequacy of two approximations to the distribution of the range in random samples from a normal population, one by Cox [J. Roy. Statist. Soc. Ser. B. 11, 101–114 (1949); these Rev. 11, 262] and the second by Patnaik [Biometrika 37, 78–87 (1950); these Rev. 12, 116] in the case of a single sample of n items for $n=4$, 6, 10, and 15. The results are applied to short cut, readily calculated tests of variance heterogeneity.

L. A. Aroian (Culver City, Calif.).

Lieblein, Julius. Properties of certain statistics involving the closest pair in a sample of three observations. J. Research Nat. Bur. Standards 48, 255–268 (1952).

The distributions of several statistics involving the closest pair of observations in a sample of three are investigated, mainly for samples from rectangular and normal populations, and the numerical values of their means and standard deviations calculated. The difference of the closest pair of observations divided by the range is shown to be a poor criterion for rejecting outlying observations. The distribution of the outlying observation is found for a rectangular population. *S. W. Nash* (Vancouver, B. C.).

✱Muniruzzaman, A. N. M. On some distributions in connection with Pareto's law. Proceedings of the First Pakistan Statistical Conference held in the University of the Panjab, Lahore, 1950, pp. 90–93. Panjab University Press, Lahore, 1951.

The author considers the estimation of and testing of hypotheses concerning the parameter ν in the Pareto distribution $(\nu-1)\omega^{\nu-1}x^{-\nu}$ $(\omega\leqq x)$, where ω is taken as known. If $x_1, \cdots, x_N$ constitute a random sample from a Pareto population, then the maximum likelihood estimate of ν is $n=1+1/[\log (G/\omega)]$, where G is the geometric mean of $x_1, \cdots, x_N$. The distribution of n is found by the method of characteristic functions; by consideration of the moments, n is shown to be asymptotically normally distributed with mean ν and standard deviation $(\nu-1)/\sqrt{N}$. For testing hypotheses, a statistic whose variance is independent of ν would be desirable; $\eta=\log (n-1)$ has variance $1/N$. The distribution of η is found, asymptotically, η is distributed with mean $\log (\nu-1)$ and variance $1/N$. *K. Arrow.*

Adam, Adolf. Reproduktive Systeme und ihre Anwendungen in der technischen Statistik. Statist. Vierteljschr. 3, 55–70 (1950).

By means of a complicated symbolism the author obtains a number of standard results, pertaining to the hypergeometric distribution, as well as a few new ones. Applications are given to elementary quality control problems.

D. G. Chapman (Seattle, Wash.).

Rosenbaum, S. The variance of least-square estimates under linear restraints. J. Roy. Statist. Soc. Ser. B. 13, 250–255 (1951).

Daniels, H. E. Note on Durbin and Stuart's formula for $E(r_s)$. J. Roy. Statist. Soc. Ser. B. 13, 310 (1951).

James, G. S. Notes on a theorem of Cochran. Proc. Cambridge Philos. Soc. 48, 443–446 (1952).

Let $q_1, \cdots, q_k$ be quadratic forms in independent standard normal variates $x_1, \cdots, x_n$, of ranks $n_1, \cdots, n_k$, and satisfying $\sum q_j=\sum x_i^2$. Then the propositions "$\sum n_j=n$", "Each q_j is a χ^2 variate", "The q_j's are distributed independently", and "Each q_j has n_j d.f." are all true if one of the first three of them is true. An example of additive dependent χ^2's is given. *M. Sandelius* (Uppsala).

Buckland, William R. A review of the literature of systematic sampling. J. Roy. Statist. Soc. Ser. B. 13, 208–215 (1951).

Daniels, H. E. The covering circle of a sample from a circular normal distribution. Biometrika 39, 137–143 (1952).

The covering circle of a set of points is the smallest circle which covers the points. An expression is obtained for the joint distribution of r and ρ, where r is the radius of the covering circle of a sample from a circular normal distribution, and ρ is the distance from the center of the covering circle to the center of the distribution. The distribution of r is expressed in terms of the best distribution, providing confidence intervals for the scale parameter.

J. L. Hodges, Jr. (Berkeley, Calif.).

Bartlett, M. S. A sampling test of the χ^2 theory for probability chains. Biometrika 39, 118–121 (1952).

In a previous paper [Proc. Cambridge Philos. Soc. 47, 86–95 (1951); these Rev. 12, 512] the author gave a χ^2 theory for Markov chains. In the present paper he applies this theory to an empirical chain with two states, obtained by use of a table of random numbers, obtaining reasonable agreement. *J. L. Doob* (Urbana, Ill.).

Rao, C. Radhakrishna. On statistics with uniformly minimum variance. Science and Culture 17, 483–484 (1952).

Ogawa, Junjiro. On a confidence interval of the ratio of population means of a bivariate normal distribution. Proc. Japan Acad. 27, 313–316 (1951).

All parameters of the population, from which a random sample is drawn, are assumed to be unknown. The maximum likelihood estimator of the ratio λ of the means (m_1, m_2) is presented together with the estimator's density function. Using tangents to a confidence ellipse for (m_1, m_2) and assuming m_1, $m_2>0$, the author obtains what he terms ". . . a confidence interval for λ in a certain sense." This is constructed in an interesting way, but it does not seem to be a confidence interval in general since the ". . . level of significance . . ." must be chosen so large that the ". . . ellipse lies wholly in the first quadrant . . ." of the (m_1, m_2)-plane. An illustrative example is included.

D. F. Votaw, Jr. (New Haven, Conn.).

Hemelrijk, J. A theorem on the sign test when ties are present. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 322–326 (1952).

When ties are present two modifications of the sign test have been proposed: (1) omit ties; (2) count half of the ties as pluses, half as minuses. The hypothesis tested is that the probability of a plus equals that of a minus. It is rejected when the number of pluses lies too far in either tail. The author proves that, for comparable levels of significance, the first modification gives at least as powerful a test as the second modification and is to be preferred. *S. W. Nash.*

Tiago de Oliveira, J. On the problem of statistical estimation. Anais Fac. Ci. Porto **35**, 229–240 (1951). (Portuguese. English summary)

If X is a random variable with distribution function $F(X|\theta)$ we state that the confidence interval $\delta(X)=(\underline{\theta}, \bar{\theta})$ covers the true value of θ with confidence coefficient α when $\Pr\{\theta\epsilon\delta(X)|\theta\}=\alpha$. The author determines intervals $\delta(X)$ such that, for a fixed X, (i) $\bar{\theta}-\underline{\theta}$ and (ii) $\log\bar{\theta}-\log\underline{\theta}$, respectively, are minimized. Fixing X, however, invalidates the concept of "shortness" of $\delta(X)$ which is the objective of the analysis [see, e.g., Neyman, Lectures and conferences on mathematical statistics, U. S. Dept. Agriculture, Washington, D. C., 1938]. *H. L. Seal* (New York, N. Y.).

Marriott, F. H. C. Tests of significance in canonical analysis. Biometrika **39**, 58–64 (1952).

In the study of the relationship between two sets of variables $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_q$ $(p\leqq q)$, the canonical correlation coefficients have been extensively used. In this paper the exact or nearly exact distributions of the greatest canonical correlation coefficients are worked out for a number of different cases of p and q, under the hypothesis that the parent canonical correlations are all zero. An approximate test is also shown to agree satisfactorily with the exact test over the range of values of p and q for which the exact test has been derived. Some tabulations of 1% and 5% significance points are given. *D. G. Chapman.*

Chernoff, Herman, and Scheffé, Henry. A generalization of the Neyman-Pearson fundamental lemma. Ann. Math. Statistics **23**, 213–225 (1952).

f_i and g_j are real integrable functions on a Euclidean space X, $y_i(S)=\int_S f_i dx$, $z_j(S)=\int_S g_j dx$ for every Borel set S $(i=1, \cdots, m; j=1, \cdots, n)$; ϕ is a function of m real variables, A is a given subset of Euclidean n-space Z, c_i are given constants, $z(S)=(z_1(S), \cdots, z_n(S))$. A simple condition is given which under certain restrictions is necessary and (under slightly different ones) sufficient for a set S to maximize $\phi(z(S))$ subject to $z(S)\epsilon A$ and $y_i(S)=c_i$ $(i=1, \cdots, m)$. This generalizes the result of Dantzig and Wald [same Ann. **22**, 87–93 (1951); these Rev. **12**, 622] for $n=1$, $\phi(z_1)=z_1$, $A=Z$. An application to the theory of Type D critical regions is given. A technique for computing the maximizing S is discussed. *J. Kiefer* (Ithaca, N. Y.).

Durbin, J., and Stuart, A. Inversions and rank correlation coefficients. J. Roy. Statist. Soc. Ser. B. **13**, 303–309 (1951).

Grüneberg, Hans-Joachim. Die multiple Faktoranalyse. Mitteilungsblatt Math. Statist. **4**, 9–31 (1952).

This is an expository article on Multiple Factor Analysis and follows closely L. L. Thurstone's Multiple-factor analysis [Univ. Chicago Press, 1947; these Rev. **9**, 47]. *E. Lukacs* (Washington, D. C.).

Tocher, K. D. On the concurrence of a set of regression lines. Biometrika **39**, 109–117 (1952).

Hypothesis H_0, the "concurrence", is that the regression lines have the same intercept on the axis of the independent variable x. Being a composite hypothesis, H_0 is treated by a maximum principle used by Barnard [Biometrika **34**, 123–138 (1947); these Rev. **8**, 395], which leads to an analysis of variance with linear constraints. When the x-observations are the same in the different regressions, the solution is obtained as an explicit F ratio. For the general case an iterative solution is indicated. Application to a colour-temperature experiment. *H. Wold* (Uppsala).

Stange, K. Über das Ausgleichen einer fehlerhaften linearen Punktreihe bei korrelativer Verknüpfung der Messfehler. Mitteilungsblatt Math. Statist. **4**, 48–70 (3 plates) (1952).

Survey of German contributions 1920–1950 to the "error-in-variable" approach, without any references to other literature [cf. the preceding review and the survey by Lindley, Suppl. J. Roy. Statist. Soc. **9**, 218–244 (1947); these Rev. **9**, 363]. Excellent graphs, which, for one thing, bring out a limitation not always stressed, viz., that when estimating the individual error-free observations in this approach the estimation error will not tend to zero with increasing number of observations. *H. Wold* (Uppsala).

Elfving, G. Optimum allocation in linear regression theory. Ann. Math. Statistics **23**, 255–262 (1952).

The author considers the situation where observations may be taken from a number of distinct information sources, and where each observation is the sum of a known linear function (particular to the source) of unknown parameters, and a random error (particular to the observation). The problem is to so distribute the observations among the various sources that the resulting parameter estimates have least variance, the total number of observations being fixed. An ingenious geometrical argument is used to show that, for the case of one or two parameters, only two or three of the potential sources of information are required. The same argument gives the optimum allocation among these sources. A general algebraic solution is also given, in which the allocation weights appear as the elements of a certain eigenvector. *P. Whittle* (Uppsala).

Grant, Alison M. Some properties of runs in smoothed random series. Biometrika **39**, 198–204 (1952).

With reference to Kermack and McKendrick [Proc. Roy. Soc. Edinburgh **57**, 228–240 (1937)] the monotonic sections (=runs) in the series $y_t=(x_t+\cdots+x_{t+a})/a$, where x_t, $x_{t+1}, \cdots$ is a purely random series, are studied with special regard to the mean amplitude of the runs as a function of their length. *H. Wold* (Uppsala).

Baten, William Dowell. Variances of differences between means when there are two missing values in randomized block designs. Biometrics **8**, 42–50 (1952).

Daniels, H. E. The theory of position finding. J. Roy. Statist. Soc. Ser. B. **13**, 186–199; discussion: 199–207 (1951).

Mathematical Biology

Rapoport, Anatol. Contribution to the mathematical theory of mass behavior. I. The propagation of single acts. Bull. Math. Biophys. **14**, 159–169 (1952).

The "single act" is one which is performed at most once by any individual. Equations are derived for the cases when the performance is induced by some external stimulus, randomly distributed in time; that in which it is induced by seeing someone else perform the act; that in which either is effective; and that in which seeing an increasing number of persons is effective rather than seeing a single one. *A. S. Householder* (Oak Ridge, Tenn.).

Rashevsky, N. Prolegomena to a dynamics of ideologies. Bull. Math. Biophys. 14, 95–118 (1952).

Ideology is defined as a verbalizable behavior pattern which may be adopted by society. Different kinds of ideologies may be represented by behavior matrices, introduced in the author's book, Mathematical biology of social behavior [Univ. Chicago Press, 1951; these Rev. 13, 370]. A uniparametric representation of such matrices is suggested and discussed. The previous results on social imitation are extended to the case of n different behaviors, each of which is determined by a particular value of a continuously varying parameter. It is shown that, depending on some other social parameters, changes from one ideology to another may proceed either quasi-continuously or definitely discontinuously. The paper concludes with some general speculations on the possibility of applying the above results to a mathematical interpretation of history. (From the author's summary.)

I. M. H. Etherington (Edinburgh).

Ogawa, Junjiro. On Weinberg's statistical method in human heredity. Proc. Japan Acad. 27, 527–531 (1951).

The problem is to estimate the probability p, with confidence limits, of the incidence of a disease in a population which consists of known victims and their siblings. A maximum likelihood estimate is obtained.

A. S. Householder (Oak Ridge, Tenn.).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. I. Distribution of genes. Proc. Japan Acad. 27, 371–377 (1951).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. II$_1$. Cross-breeding phenomena. Proc. Japan Acad. 27, 378–383 (1951).

Komatu, Yûsaku. Probability-theoretic investigations on inheritance. II$_2$. Cross-breeding phenomena. Proc. Japan Acad. 27, 384–387 (1951).

The genetic equilibrium of mixtures of populations in genetic equilibrium is studied for multiple alleles at one diploid locus. For example, Hardy's second-generation theorem is extended, and it is shown that the frequency of each type of homozygote is never increased and is typically decreased by the mixtures in question. *L. J. Savage.*

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. X. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 78–80 (1952).

The integrand in the variational problem is set up so as to allow for interaction with the soma.

A. S. Householder (Oak Ridge, Tenn.).

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. XI. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 264–266 (1952).

In continuation of the formal development, expressions taking account of the interaction of the soma and the chromosomes are given. *A. S. Householder.*

TOPOLOGY

Luce, R. Duncan. Two decomposition theorems for a class of finite oriented graphs. Amer. J. Math. 74, 701–722 (1952).

A network, as defined by the author, is a graph in which each edge is taken with a definite direction and no two vertices are joined by two edges having the same direction. A network is connected if it is possible to pass from any vertex to any other following always the directions of the edges. The idea of connection leads to the following definitions. The degree of a network N is the least number of edges whose removal leaves a disconnected network. N is k-minimal if the removal of an edge always gives a network of degree $k-1$ and k-uniform if every complete connected subnetwork is of degree $\leqq k$. (A complete subnetwork of N is made up of all the vertices and some of the edges of N.) A connected 1-minimal network is called minimal.

The author's first decomposition theorem states that any network N can be expressed as a union of complete 1-minimal subnetworks, no two of which have a common edge. Other conditions may be imposed on these subnetworks. For example, we can arrange that just one is disconnected and that the others are 1-uniform. Minimal networks correspond to the trees of ordinary graph theory, but are more difficult to characterize. The author's second decomposition theorem describes directly the structure of a minimal network. Some applications are made to the theory of connected networks. *W. T. Tutte* (Toronto, Ont.).

Efremovič, V. A. Invariant definition of topological product. Uspehi Matem. Nauk (N.S.) 7, no. 1(47), 159–161 (1952). (Russian)

The following definition is given (and shown to be equivalent with the usual one) of the topological product of spaces X_α: a point $x=\{x_\alpha\}$ belongs to the closure of A if and only if, given arbitrary A_k such that $A=A_1\cup\cdots\cup A_n$, there exists A_{k_0} such that, for any α, x_α belongs to the closure of the projection of A into X_α. *M. Katětov* (Prague).

Pettis, B. J. Remarks on a theorem of E. J. McShane. Proc. Amer. Math. Soc. 2, 166–171 (1951).

E. J. McShane has recently proved a basic, purely topological theorem [Ann. of Math. 51, 380–386 (1950); these Rev. 12, 518] from which follow a number of results on topological groups, linear spaces, etc., to the effect that if a second category set satisfies the Baire condition and is endowed with some other property, then it has an interior point. This paper presents the following theorem: Let X_1 and X_2 be topological spaces and $\mathfrak{F}$ be a non-void collection of functions f, each of which has an open domain D_f in X_1 and maps open sets onto open sets in X_2. Then $\bigcup_{f\epsilon\mathfrak{F}} f(N\cap S\cap D_f)$ is non-void and open provided that N is a non-void open set of $(\operatorname{int}(X_1-I(S)))\cap I(X_1-S)$ (where $I(A)=\bigcup[G\,|\,G \text{ open}, G\cap A \text{ is first category}]$), that $D_f\supset S\cap N$ for every $f\epsilon\mathfrak{F}$, and that

$$y\epsilon\bigcup_{f\epsilon\mathfrak{F}} f(N\cap D_f) \text{ implies } N\cap\left(\bigcup_{f\epsilon\mathfrak{F}} f^{-1}(y)\right)$$

is second category. A stronger and more specific version of McShane's theorem is obtained from this result by letting $N=\operatorname{int}(X_1-I(S))$, where S is taken to be a Baire set. Improved versions of some other results contained in McShane's paper are also derived. This paper also gives an extension of Zorn's theorem on the structure of a semi-group S (where S is a Baire set such that S and S^{-1} are second category at e) [cf. Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., vol. 31, New York, 1948, pp. 156–158; these Rev. 9, 594; and McShane, loc. cit.].

H. Tong (New York, N. Y.).

Balanzat, Manuel. On the metrization of quasi-metric spaces. Gaz. Mat., Lisboa 12, no. 50, 91–94 (1951). (Spanish)

The author calls a space quasi-metric provided there is attached to each ordered element-pair x, y a non-negative real number xy such that $xy=0$ if and only if $x=y$ and $xy \leqq xz+zy$, while the fundamental system of neighborhoods of an element p is the set of all open spheres with center p. The paper gives an example of a completely normal quasi-metric space which is not metrizable, and hence answers in the negative a question raised by Ribeiro. It also exhibits (1) a compact quasi-metric space with a denumerable basis and (2) a Hausdorff quasi-metric space with denumerable basis, neither of which is metrizable. Nor is a completely regular quasi-metric space necessarily metrizable. Leaning heavily on a theorem of Niemytzki [Trans. Amer. Math. Soc. 29, 507–513 (1927)] it is shown that a quasi-metric space E is metrizable if for each closed subset A and each element p not belonging to A there is definable in E a uniformly continuous (numerical) function f with range in the interval (0, 1) and such that $f(p)=0$ and $f(x)=1$ for every x in A. *L. M. Blumenthal* (Los Angeles, Calif.).

Monteiro, A. A. Les filtres fermés des espaces compacts. Gaz. Mat., Lisboa 12, no. 50, 95–96 (1951).

If R is a lattice, let Φ_R be the lattice of all filters (i.e. dual ideals) of R. The author has established the following characterization of Φ_F as a lattice, where F is the lattice of all closed sets of some compact Hausdorff space I. In order that a lattice Φ be isomorphic to some Φ_F it is necessary and sufficient that: 1) Φ be distributive, complete and compact ($x \varepsilon \Phi$ is "compact" if whenever $\bigcap_{\alpha \varepsilon A} x_\alpha \subseteq x$ there is a finite set $B \subseteq A$ such that $\bigcap_{\beta \varepsilon B} x_\beta \subseteq x$; Φ is "compact" if its first element 0 is compact); 2) every element of Φ is the infimum of compact elements; 3) the compact elements of Φ different from 0 are precisely the atomic elements ($x \varepsilon \Phi$ is "atomic" if it is the supremum of some non-empty family of atoms in the usual sense); 4) Φ is normal in the sense that, if $a, b \varepsilon \Phi$, $a \cap b = 0$, there are $a_1, b_1 \varepsilon \Phi$ such that $a \cap b_1 = a_1 \cap b = 0$, $a_1 \cup b_1 = 1$. The author also indicates how these conditions should be slightly modified in order that the base space I be metrizable, or totally disconnected or finite. [Related results are to be found in Birkhoff and Frink, Trans. Amer. Math. Soc. 64, 299–316 (1948); Nachbin, Fund. Math. 36, 137–142 (1949); these Rev. 10, 279; 11, 712.] *L. Nachbin* (Rio de Janeiro).

Ellis, David. On the metric characterization of metric lattices. J. Indian Math. Soc. (N.S.) 15 (1951), 152–154 (1952).

V. Glivenko [Amer. J. Math. 58, 799–828 (1936)], and Smiley and Transue [Bull. Amer. Math. Soc. 49, 280–287 (1943); these Rev. 4, 248] proved that a metric space M is congruent to a metric lattice with first element if and only if M is "almost ordered" with respect to one of its elements. This proposition is completed here by Th. I: Every compact lattice possesses a first (and last) element f. The proof consists in considering a sequence $p_1, \cdots, p_n, \cdots$ metrically everywhere dense, then the sequence $f_n =$ meet of $p_1, \cdots, p_n$ and defining f as an element of accumulation of the f_n. Th. II: Every compact metric lattice is lattice-complete. This assertion follows readily from the property for a metrically complete metric lattice to be conditionally lattice-complete. *C. Y. Pauc* (Cape Town).

Burgess, C. E. Continua and their complementary domains in the plane. II. Duke Math. J. 19, 223–230 (1952).

[For part I see same J. 18, 901–907 (1951); these Rev. 13, 484.] Conditions on complementary domains are given which require a plane continuum to satisfy some indecomposability condition. The author proves that if M is a continuum and for each two domains intersecting M there exist three complementary domains of M each intersecting each of the given domains, then there is a positive integer n less than five such that M is the finished sum of n indecomposable continua. *H. M. Gehman* (Buffalo, N. Y.).

Edelstein, Michael. On the non-decomposability of the plane by a Jordan arc. Riveon Lematematika 5, 49–52 (1952). (Hebrew. English summary)

The author proves the non-decomposability of the plane by a Jordan arc basing his proof on Jordan's theorem for polygons only. *S. Agmon* (New York, N. Y.).

Jackson, James R. Some theorems concerning absolute neighborhood retracts. Pacific J. Math. 2, 185–189 (1952).

The author proves among other results the following: A space Y is an ANR if and only if it has countably many components, each open and each a separable (metric) ANR. Let Y be an ANR and let X be compact (not necessarily metric). Then $Y^X\{X_0, y_0\}$ has open arcwise components. Let also X be regular or locally separable with an infinite basis of cardinal a. Then the homotopy classes of functions in $Y^X\{X_0, y_0\}$ has cardinal at most a. *A. D. Wallace.*

Brouwer, L. E. J. An intuitionist correction of the fixed-point theorem on the sphere. Proc. Roy. Soc. London. Ser. A. 213, 1–2 (1952).

From the intuitionist point of view the fixed-point theorem for the sphere is as invalid as the Bolzano-Weierstrass theorem. However, given a topological transformation of the sphere into itself of degree 1 one can find points which undergo arbitrarily small displacements.

H. Freudenthal (Utrecht).

Dowker, C. H. Topology of metric complexes. Amer. J. Math. 74, 555–577 (1952).

The author defines the concepts of affine complex, topological complex, and metric complex. For affine complexes J. H. C. Whitehead [Proc. London Math. Soc. (2) 45, 243–327 (1939)] has defined a topology which makes the complex into a topological complex which is not, in general, a metric complex. In the present paper the author studies metric complexes by comparing them with the same complex regarded as a topological complex with the Whitehead topology. In particular, he proves that any metric complex has the same homotopy type as the corresponding Whitehead complex. It follows that isomorphic metric complexes have the same homotopy type and that the singular homology groups of a metric complex are isomorphic to the combinatorial homology groups. *E. Spanier* (Chicago, Ill.).

Pannwitz, Erika. Eine freie Abbildung der n-dimensionalen Sphäre in die Ebene. Math. Nachr. 7, 183–185 (1952).

H. Hopf [Fund. Math. 28, 33–57 (1937)] has defined a mapping f of a topological space T into a space T' to be free if there exists a map g of T into T with $fg(x) \neq f(x)$ for all

$x \epsilon T$. Among the results of Hopf are the following: for $n \geqq 1$, there exists no free mapping of the n-dimensional sphere S^n into 1-dimensional Euclidean space R^1; for each $n \geqq 3$, there exists a free mapping of S^n into R^n. The author here proves that for each $n \geqq 0$ there exists a free mapping of S^n into the plane R^2. *E. E. Floyd* (Charlottesville, Va.).

Wada, Hidekazu. Über eine Vereinigung der Sätze von H. Hopf und N. Bruschlinsky. Tôhoku Math. J. (2) 4, 77–79 (1952).

A slight generalization of the Hopf classification theorem is given by replacing the role of the n-sphere S_n ($n \geqq 2$) by a connected topological space Y_n and a basic point $y_0 \epsilon Y_n$ such that $\pi_k(Y_n)=0$ ($2 \leqq k < n$), $\pi_1(Y_n)$ is either 0 or free cyclic, and some other quite complicated conditions. Obviously, S_n is a Y_n; however, the author gives no familiar spaces other than S_n which satisfy these conditions. If K is an n-dimensional locally finite simplicial complex and L a subcomplex of K, then the generalized theorem states as follows: The homotopy classes relative to L of the maps $Y_n^K(L, y_0)$ are in a one-to-one correspondence with the direct sum

$$H^1(K-L, \pi_1(Y_n)) \oplus H^n(K-L, \pi_n(Y_n)).$$

At the end of the paper, the author indicates that this generalized theorem can also be extended to compact normal spaces with dimension not greater than n provided that Y_n is an ANR. *S. T. Hu* (New Orleans, La.).

√ ***Fox, R. H. Recent development of knot theory at Princeton.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 453–457. Amer. Math. Soc., Providence, R. I., 1952.

Der Verf. gibt eine kurze Übersicht über die Fortschritte in der Knotentheorie, die er mit seinen Mitarbeitern in den letzten Jahren erzielt hat, und erwähnt die Methoden, die dabei entwickelt wurden; es werden keine Beweise skizziert. Die Resultate, z.T. nur angedeutet, sind noch nicht alle publiziert; die Methoden scheinen über den vorliegenden Zweck hinaus für die Topologie und die Gruppentheorie von Bedeutung zu sein. So sei hier der vom Verf. eingeführte Algorithmus (Ableitungen in beliebigen Gruppen) genannt, der es gestattet, von der Knotengruppe bezw. dem "Knotensystem" aus das Alexandersche Polynom und andere Invarianten zu erhalten und zu verallgemeinern; auch die Reidemeistersche kombinatorische Klassifikation der 3-dimensionalen Linsenräume kann in neuer Weise gewonnen werden. Zu den behandelten Fragen gehören ferner die Überlagerungen der Aussenräume von Knoten und die Krümmung eines Knotentypus. *B. Eckmann* (Zürich).

Fox, R. H. On the complementary domains of a certain pair of inequivalent knots. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 37–40 (1952).

Ein Beispiel von zwei inäquivalenten Knoten mit isomorpher Knotengruppe ist von Seifert [S.-B. Preuss. Akad. Wiss. 1933, 811–828] durch Zusammensetzen von Kleeblattschlingen konstruiert worden. In der vorliegenden Note wird gezeigt, dass die Aussenräume dieser Knoten nicht homöomorph sind. Der Verf. definiert hiezu "peripherale" Untergruppen der Fundamentalgruppe eines topologischen Raumes und betrachtet, im Falle der beiden Aussenräume, deren Verhalten bei den Darstellungen der Knotengruppe durch Permutationen von 5 Elementen. *B. Eckmann.*

Eilenberg, Samuel, and MacLane, Saunders. Cohomology groups of abelian groups and homotopy theory. IV. Proc. Nat. Acad. Sci. U. S. A. 38, 325–329 (1952).

Cette note donne un théorème d'extension et un théorème de classification pour les applications continues d'un complexe K dans un espace X, connexe par arcs, tel que $\pi_i(X)=0$ pour $i<n$ et $n<i<q$ ($1<n<q$; le cas $n=1$ est aussi envisagé pour le théorème de classification). Ces deux théorèmes englobent tous les résultats connus dans le cas $q=n+1$ [Steenrod, Whitney, Postnikov, J. H. C. Whitehead; voir aussi Eilenberg, Proc. Int. Congress Math., 1950, vol. 2, pp. 350–353; ces Rev. 13, 575; nous renvoyons le lecteur à cette analyse pour les notations ci-dessous].

Les auteurs introduisent deux nouvelles sortes de "produits": le produit relativement à un élément $y \epsilon H^{q+1}(\Pi, n; G)$ sert au théorème d'extension; le "depressed product" relativement à y sert au théorème de classification. Dans chaque cas, on prend pour y l'invariant d'Eilenberg-MacLane $\mathbf{k}^{q+1} \epsilon H^{q+1}(\pi_n, n; \pi_q)$ de l'espace X. Lorsque $q=n+1$, le groupe $H^{n+2}(\Pi, n; G)$ est explicitement connu, et en explicitant les "produits" relativement à $\mathbf{k}^{n+2}$ on retrouve les résultats connus.

Les "produits" sont définis dans les circonstances suivantes: soient K un complexe semi-simplicial, L_i ($1 \leqq i \leqq r$) des sous-complexes, L la réunion des L_i. Si $x_i \epsilon H^n(K, L_i; \Pi)$ et $y \epsilon H^{q+1}(\Pi, n; G)$, le produit $[x_1, \cdots, x_r; y]$ est dans $H^{q+1}(K, L; G)$. Pour $r=1$, $[x; y]=T^*(x)\cdot y$, où $T^*(x)$: $H^{q+1}(\Pi, n; G) \to H^{q+1}(K, L; G)$ est défini par l'application simpliciale $T(z)$: $K \to K(\Pi, n)$ relative à un cocycle z de la classe de x. L'application $T^*(x)$ n'est pas additive en x; pour r quelconque, la définition de $[x_1, \cdots, x_r; y]$ par récurrence sur r fait intervenir la déviation de $T^*(x)$ vis-à-vis de l'addition. La définition des "depressed products" est un peu plus compliquée et fait intervenir la "suspension" des groupes d'Eilenberg-MacLane. *H. Cartan* (Paris).

Steenrod, N. E. Reduced powers of cohomology classes. Ann. of Math. (2) 56, 47–67 (1952).

Ceci est le premier de deux articles consacrés à l'exposition complète de la théorie des "puissances réduites", c'est-à-dire des puissances de Steenrod qui généralisent à p quelconque les "carrés de Steenrod" relatifs à $p=2$. Des résultats préliminaires avaient été annoncés au Congrès International de 1950; des démonstrations étaient déjà connues par des notes miméographiées, grâce à quoi des résultats substantiels ont déjà été obtenus en topologie par l'utilisation des p-puissances [par ex., A. Borel et Serre, C. R. Acad. Sci. Paris 233, 680–682 (1951); ces Rev. 13, 319]. Dans le présent article, l'auteur définit et étudie des opérations plus générales que les puissances réduites, mais il ne prouve pas qu'elles donnent substantiellement autre chose que les puissances réduites. Un article ultérieur sera consacré aux propriétés spéciales des puissances réduites.

On définit d'abord l'importante notion d'"opération cohomologique": q et r étant des entiers donnés, G et G' des groupes de coefficients, une opération cohomologique associe à tout complexe K une application (non nécessairement un homomorphisme) $H^q(K; G) \to H^r(K; G')$ de manière à commuter avec les homomorphismes induits par une application d'un complexe dans un autre. L'auteur définit ensuite des opérations cohomologiques dans les circonstances suivantes: soient p un entier $\geqq 2$ donné une fois pour toutes, Π le groupe symétrique de p éléments, Γ l'algèbre de Π à coefficients dans l'anneau Z des entiers, s l'homomorphisme $\Gamma \to Z$ qui associe $+1$ à chaque élément de Π, t l'homomorphisme $\Gamma \to Z$ qui associe $+1$ à un élément de Π si c'est

une permutation paire, -1 dans le cas contraire. On appelle 0-sequence une suite d'éléments $\alpha_i \epsilon \Gamma$ ($i \geqq 1$) telle que $s(\alpha_1)=0$, $\alpha_{i+1}\alpha_i=0$. Une 0-sequence définit, quels que soient les entiers q et $i \geqq 0$, les groupes G et G' de coefficients, et l'application p-linéaire symétrique $G \times \cdots \times G \rightarrow G'$, une opération cohomologique

$$\mathcal{P}_i^p\colon \quad H^q(K;G) \rightarrow H^{pq-i}(K;G'/nG'),$$

où n désigne l'entier $s(\alpha_i)$ si q est pair, $t(\alpha_i)$ si q est impair; pour $i=0$, c'est la puissance p-ième (au sens du cup product) et $n=0$. Ces opérations sont les "puissances réduites" quand on prend la 0-sequence suivante: g est une permutation circulaire sur p éléments, $\alpha_{2i+1}=e-g$ (e: élément neutre de Π), $\alpha_{2i}=\sum_{k=0}^{p-1} g^k$. [N.B.: on sait que cette 0-sequence définit un "complexe" servant à calculer la cohomologie du groupe cyclique d'ordre p. Dans le cas général, une 0-sequence définit toujours un "complexe", dont l'introduction permet de simplifier un peu l'exposition.]

L'invariance topologique des opérations, dans le cas général, résulte facilement des constructions. [Note du rapporteur. On pourrait faire des constructions analogues sur le complexe singulier d'un espace topologique, ce qui définirait les opérations $\mathcal{P}_i^p$ pour tous les espaces topologiques. Or, d'après une remarque inédite de Serre, une construction cohomologique $H^q(X;G) \rightarrow H^r(X;G')$ définie pour tous les espaces, est connue quand on la connaît dans un espace X tel que $\pi_i(X)=0$ pour $i \neq q$, $\pi_q(X)=G$; les groupes $H^r(X;G')$ d'un tel espace sont les "groupes d'Eilenberg-MacLane" $H^r(G,q;G')$.]

L'auteur démontre une série de propriétés générales des opérations $\mathcal{P}_i^p$ associées à une 0-sequence. Notamment: $\mathcal{P}_i^p=0$ pour $pq-i<q$; sur $H^q(K;Z)$, $\mathcal{P}^p{}_{pq-q}$ est proportionnel à la réduction modulo n; sur $H^1(K;Z)$, $\mathcal{P}_i^p=0$ pour $i \neq p-1$; sur $H^2(K;Z)$, $\mathcal{P}_i^p=0$ pour i impair, $\mathcal{P}^p{}_{2k}(u)$ est proportionnel à u^{p-k} réduit modulo n. [En fait, toutes ces propriétés sont vraies de toute opération cohomologique, en vertu de la remarque de Serre.] Les autres propriétés sont particulières aux opérations $\mathcal{P}_i^p$; elles se déduisent d'un principe général donnant une condition suffisante pour que l'on ait $m\mathcal{P}_i^p=0$ pour $i \geqq k$ (m: entier convenable). Par ex., si α_1 appartient à l'algèbre d'un sous-groupe d'ordre m de Π, on a $m\mathcal{P}_i^p=0$ pour tout $i \geqq 1$.

Bien entendu, toute la théorie vaut (et est exposée) pour la cohomologie relative $H^q(K,L;G)$, L désignant un sous-complexe de K.

H. Cartan (Paris).

Yang, Chung-Tao. On cohomology theories. Proc. Nat. Acad. Sci. U. S. A. **38**, 348–351 (1952).

The author proves that for fully normal spaces the Alexander-Kolmogoroff cohomology theory is isomorphic to the unrestricted Čech cohomology theory (based on arbitrary open coverings). Inasmuch as Dowker [Ann. of Math. 51, 278–292 (1950); these Rev. 11, 450] has shown that the homotopy axiom is valid for these unrestricted Čech groups, it follows that for fully normal spaces the homotopy axiom is also valid for the Alexander-Kolmogoroff groups.

E. Spanier (Chicago, Ill.).

Heller, Alex. Singular homology in fibre bundles. Ann. of Math. (2) **55**, 232–249 (1952).

Fiber bundles are investigated by methods of simplicial singular homology theory. Many geometric concepts involved in the theory are algebraicized by passing from topological spaces to their total singular complexes and more generally to abstract semi-simplicial complexes. To carry this out the author introduces for semi-simplicial complexes analogs of many topological concepts like homotopy, topological group of operators, etc. The main result shows that within this more general setting it is possible to establish the existence of "universal fiber bundles" for any topological group G.

S. Eilenberg (New York, N. Y.).

Whitehead, George W. Fiber spaces and the Eilenberg homology groups. Proc. Nat. Acad. Sci. U. S. A. **38**, 426–430 (1952).

In this note the author proves that for any arc-wise connected topological space X, there exists for each positive integer n a fiber mapping $p\colon T \rightarrow X$ such that (1) the homotopy groups $\pi_q(T)$ are trivial for $q \leqq n$, and (2) p induces isomorphisms of $\pi_q(T)$ onto $\pi_q(X)$ for $q>n$. Here the term "fiber mapping" is to be understood in the sense defined by Serre [Ann. of Math. **54**, 425–505 (1951); these Rev. **13**, 574]. This theorem solves a problem proposed by Hurewicz in 1947 [cf. problem 32 in the list given by Eilenberg, ibid. **50**, 247–260 (1949); these Rev. **10**, 726].

The author uses this result to study the so-called Eilenberg homology groups of X, defined as follows. Choose a basepoint $x \epsilon X$, and for each integer n, let $S_n(X)$ be the subcomplex of the total singular complex $S(X)$ consisting of all singular simplexes whose n-dimensional skeletons are mapped into x. The Eilenberg homology groups of X are then defined to be the homology groups of the complexes $S_n(X)$. The main result is that if $q \leqq 2n$, then the relative homology group $H_q(S_{n-1}(X), S_n(X))$ is naturally isomorphic to the q-dimensional homology group of the Eilenberg-MacLane complex, $K[\pi_n(X), n]$.

In a note added in proof, the author observes that the results of this paper have been independently discovered by H. Cartan and J. P. Serre [C. R. Acad. Sci. Paris **234**, 288–290, 393–395 (1952); these Rev. **13**, 675].

W. S. Massey (Providence, R. I.).

Dowker, C. H. Homology groups of relations. Ann. of Math. (2) **56**, 84–95 (1952).

Let R denote a relation between the elements of a set X and the elements of a set Y. Let K be the simplicial complex whose simplices are those finite subsets of X which are R-related to some common element of Y and let L be the complex whose simplices are the finite subsets of Y which are R-related to some element of X. By defining in a natural way simplicial mappings from the first barycentric subdivision K' of K to L and from L' to K the author shows that K and L have isomorphic homology and cohomology groups.

Given a space and a covering of the space there is the relation of a point of the space belonging to the covering. One of the complexes associated with this relation is the nerve of the covering and is used in defining the Čech homology and cohomology groups of the space. The other complex associated with the relation is used in defining the Vietoris homology groups and the Alexander cohomology groups of the space. Using the fact that these complexes have isomorphic homology and cohomology groups and the further fact that the isomorphism between them commutes with mappings of relations, it is proved that the Čech and Vietoris homology groups and the Čech and Alexander cohomology groups of a space are isomorphic when based on the same family of coverings. These isomorphisms are shown to commute with the induced homomorphisms and boundary and coboundary operators of each theory. Since the Čech theory based on all open coverings satisfies the

Eilenberg-Steenrod axioms, it follows that the Alexander cohomology theory does also. *E. H. Spanier.*

Morse, Marston. Homology relations on regular orientable manifolds. Proc. Nat. Acad. Sci. U. S. A. **38**, 247–258 (1952).

In 1927, Morse stated certain theorems [same Proc. **13**, 813–817 (1927)] relating the critical points of a non-degenerate function f on a regular n-manifold Σ in a euclidean space to the connectivity numbers of Σ and certain of its subspaces of the forms $f=a$, $f\leqq a$, etc. He here proves these theorems, with certain extensions, since more recent topological developments have been based on them. The principal extension consists in replacing chains over the integers mod 2 by chains over a general field.

S. S. Cairns (Urbana, Ill.).

GEOMETRY

Thébault, Victor. Sur la géométrie du triangle et du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 5–12 (1952).

Thébault, Victor. A propos du tranchet d'Archimède. Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 41–48 (1952).

Gaddum, J. W. The sums of the dihedral and trihedral angles in a tetrahedron. Amer. Math. Monthly **59**, 370–371 (1952).

The author shows, in an elementary way, that the sum of the dihedral angles of a tetrahedron is comprised between 2π and 3π, and consequently the sum of the trihedral angles of a tetrahedron lies between 0 and 2π. Moreover, these bounds are the best that can be given. *N. A. Court.*

Marmion, Alphonse. Extension de la notion d'orthopôle. C. R. Acad. Sci. Paris **234**, 2420–2422 (1952).

The author shows that the established notion and the main properties of the orthopole of a plane with respect to a tetrahedron, that is, with respect to four planes, may, with slight modifications, be applied to four lines in three-dimensional space. This new application, moreover, may be extended to an Euclidean space of n dimensions. This note supplements a previous note by the same author [same C. R. **234**, 2040–2042 (1952); these Rev. **13**, 861].

N. A. Court (Norman, Okla.).

✓***Zühlke, Paul. Konstruktionen in begrenzter Ebene.** 3d ed. B. G. Teubner Verlagsgesellschaft, Leipzig, 1951. 42 pp. 2.10 DM.

✓***Lietzmann, W. Altes und Neues vom Kreis.** 2d ed. B. G. Teubner Verlagsgesellschaft, Leipzig, 1951. 55 pp. 2.10 DM.

Fempl, Stanimir. Über den Zentriwinkel der Abwicklung des Mantels eines schiefen Kreiskegels. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7**, 30–35 (1952). (Serbo-Croatian. German summary)

Lorent, H. Sur un ou deux ensembles de circonférences du plan ou de sphères de l'espace. Bull. Soc. Roy. Sci. Liège **21**, 24–39 (1952).

Weber, Werner. Der Hauptsatz über apolare Kurven. Collectanea Math. **4**, 71–82 (1951).

In an earlier paper [Collectanea Math. **3**, 121–135 (1950); these Rev. **13**, 153] the author enumerates six degenerate cases of apolarity. He now finds, in each case, which points on the conic O can be used to build up a triangle inscribed in O and self-polar for the other given conic K.

H. S. M. Coxeter (Toronto, Ont.).

Shephard, G. C. Regular complex polytopes. Proc. London Math. Soc. (3) **2**, 82–97 (1952).

The author extends the concept of a polytope from real to complex Euclidean space, using a unitary metric so that the square of the distance between points $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ is $\sum(x_i-y_i)(\bar{x}_i-\bar{y}_i)$. A polytope is a finite, connected configuration of points, lines, planes, etc., and is said to be regular if its symmetry group is transitive on the elements of each kind, while certain other conditions are satisfied as well. In particular, a one-dimensional regular polytope, called an m-set, consists of m collinear points whose abscissas are the powers of a primitive mth root of unity. One requirement for an n-dimensional regular polytope ($n>1$) is that the lines through each point contain k equal m-sets such that the remaining $k(m-1)$ points of these sets lie in $m-1$ parallel hyperplanes, each containing k points, one from each m-set. The $(n-1)$-dimensional configuration formed by such a set of k points is called the vertex figure. In particular, a two-dimensional regular polytope is a configuration of points and lines, say vertices and edges, with the vertices lying also on certain other lines, the vertex figures. The remaining requirement in this case ($n=2$) is the existence of symmetry operations which cyclically permute the points on each line of either type. For $n>2$, the definition is inductive: the $(n-1)$-dimensional sub-configurations and the vertex figures must be regular. The author gives a complete enumeration of the regular polytopes so defined, using the following notation.

✓ If a two-dimensional regular polytope has q/p_1 edges, these being p_1-sets, while the vertex figure is a p_2-set, then the number of vertices is q/p_2 and an appropriate "Schläfli symbol" is $p_1(q)p_2$. If a three-dimensional regular polytope has a $p_1(q)p_2$ in each of its planes, while the vertex figure is a $p_2(q_2)p_3$, an appropriate symbol is $p_1(q_1)p_2(q_2)p_3$. Similarly in n dimensions we "telescope" the symbols for the $(n-1)$-dimensional sub-configuration and vertex figure to obtain $p_1(q_1)p_2\cdots(q_{n-1})p_n$. Every such polytope has a "reciprocal", $p_n(q_{n-1})\cdots p_2(q_1)p_1$. In particular, $2(2p)2$ is the ordinary p-gon, and $2(2p)2(2q)\cdots 2$ is the real polytope $\{p, q, \cdots\}$ [Coxeter, Regular polytopes, Methuen, London, 1948, p. 129; these Rev. **10**, 261]. The octahedron $2(6)2(8)2$ is a special case of the "generalized cross polytope" $2(6)2\cdots(6)2(2m^2)m$, which has mn vertices [Coxeter, same Proc. (2) **41**, 278–301 (1936), pp. 286–287, with m and n interchanged].

He observes that, when the real and imaginary parts of the coordinates of the vertices are interpreted as coordinates in real Euclidean space of twice as many dimensions, a corresponding $2n$-dimensional real polytope (not necessarily regular) is obtained. In this manner the 3- and 4-dimensional complex polytopes

$$3(24)3(18)2,\quad 3(24)3(24)3 \quad\text{and}\quad 3(24)3(24)3(24)3$$

yield the 6- and 8-dimensional real polytopes 1_{22}, 2_{21} and

4_{21} [Coxeter, Amer. J. Math. 62, 457–486 (1940), pp. 469–480; these Rev. 2, 10]. Similarly, the generalized orthotope $m(2m^2)2(6)2\cdots(6)2$, reciprocal to the generalized cross polytope, yields the rectangular product of n m-gons [Coxeter, Regular Polytopes, p. 124]. Every regular polytope has a center, and the vertices usually fall into sets of two or more lying on lines through the center. These "diameters" may be interpreted as the points of a configuration in complex projective $(n-1)$-space. It is significant that several well-known configurations arise in this manner.
H. S. M. Coxeter (Toronto, Ont.).

Christov, Chr. Sur les distances entre les points d'un espace euclidien ou pseudo-euclidien. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1, 46, 9–20 (1950). (Bulgarian. French summary)

The author considers a space of ordered m-tuples $(y^1, y^2, \cdots, y^m)$ of complex numbers with the squared-distance between two points y_r, y_s given by

$$\sum_{a,b=1}^{m} g_{a,b}(y_r^a-y_s^a)(y_r^b-y_s^b).$$

He obtains necessary and sufficient conditions in order that a set c_{rs} of real numbers, $c_{rr}=0$; $c_{rs}=c_{sr}$ $(r, s=0, 1, \cdots, n)$ be the squared-distances of real, complex $(n+1)$-tuples of such a space. *L. M. Blumenthal* (Los Angeles, Calif.).

Christov, Chr. Une relation entre les volumes d'un simplexe et de ses simplexes limites. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 46, 21–30 (1950). (Bulgarian. French summary)

Let p_i, p_j, p_k be three of the n vertices $(n\geqq 3)$ of a simplex whose squared-volume is denoted by U. Denote by U_i, U_j, U_k the squared-volumes of the sub-simplices obtained by suppressing p_i, p_j, p_k, respectively, and define in like manner the symbols U_{ij}, U_{jk}, U_{ik}, U_{ijk}. The author establishes a complicated homogeneous relation of order 4 in these eight quantities. *L. M. Blumenthal* (Los Angeles, Calif.).

✓**★Zacharias, Max. Einführung in die projektive Geometrie.** 4th ed. B. G. Teubner Verlagsgesellschaft, Leipzig, 1951. 54 pp. 2 DM.

Di Noi, Salvatore. Sul significato proiettivo della distanza tra due punti del piano. Period. Mat. (4) 30, 79–97 (1952).

Kretschmer, Annemarie. Die projektiven Involutionen des n-dimensionalen Raumes. J. Reine Angew. Math. 186, 241–253 (1949).

The involutionary character of a projective transformation of the elements of a R_n is determined by a certain number of involutionary pairs of points. For a line one pair already suffices, for a plane two involutionary pairs of which no three points are lying on a line are needed. In this paper the general case is studied. The cases for n up to 5 are dealt with in detail. *H. A. Lauwerier* (Amsterdam).

Spampinato, Nicolò. Le tre rappresentazioni reali del piano proiettivo complesso e delle catene tridimensionali. Ricerca, Napoli 1, no. 1, 6–14, no. 2–3, 3–9, no. 4, 9–16 (1950); 2, no. 1, 1–8, no. 2, 3–10 (1951).

Die drei reellen Darstellungen der Punkte einer complexen projectiven Geraden nach v. Staudt, Riemann, Gauss-Argand werden auf die complexe projective Ebene verallgemeinert. Dies führt im 1. Falle auf eine elliptische Congruenz Γ von ∞^4 reellen Geraden des S_5. Im 2. Falle ergibt sich ausser der Darstellung auf einer Hypersphäre des S_4 noch eine Darstellung auf einer 4-dimensionalen rationalen Segre'schen Mannigfaltigkeit der Ordnung 6 des S_8: Die Riemann'sche Hypersphäre erweist sich als eine Projection dieser Mannigfaltigkeit von einem S_2 aus auf einen passend gewählten S_5 des S_8. In den verschiedenen Arten der Abbildung der complexen Ebene auf reelle höherdimensionale Gebilde werden ausführlich die Darstellungen der Geraden der complexen Ebene und die Darstellung der v. Staudt'schen Ketten, insbesondere der Dimensionen 1 und 2, untersucht. *R. Moufang* (Frankfurt a.M.).

Spampinato, Nicolò. Sulle estensioni della geometria dei raggi reciproci del Klein. Ricerca, Napoli 2, no. 3–4, 10–13 (1951).

Nach den Ergebnissen der vorstehend referierten Arbeit lässt sich der complexe projective S_r im Reellen darstellen durch eine Congruenz Γ von reellen Geraden des S_{2r+1}. Die Gruppe der Projectiven Abbildungen des S_{2r+1}, die Γ in sich überführen, induziert auf Γ eine Gruppe von Transformationen, die Bild sind von einer gewissen Transformationsgruppe des complexen S_r; diese Gruppe heisst die Gruppe der Körperprojectivitäten des S_r. Die durch diese Gruppe bestimmte Geometrie des complexen S_1 erweist sich als äquivalent mit der Geometrie der reziproken Radien von Felix Klein. Jede Körperprojectivität des S_1 transformiert in sich die Gesamtheit der v. Staudt'schen Ketten des S_1. Analoge Sätze gelten für die Ketten maximaler Dimensionszahl des complexen projectiven S_r. Die Körperprojectivitäten des complexen S_2 werden in den drei reellen Darstellungen der complexen Ebene in verschiedener Weise dargestellt. In der 3. Darstellung des complexen projectiven S_r $(r\geqq 2)$ ist eine Gruppe von quadratischen Transformationen das Bild der Körperprojectivitäten des S_r.
R. Moufang (Frankfurt a.M.).

★Nestorovič, N. M. Geometričeskie postroeniya v ploskosti Lobačevskogo. [Geometric constructions in the Lobačevskiĭ plane.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 304 pp. 10.30 rubles.

The preface to the book states correctly that familiarity with hyperbolic geometry can only be acquired by operating with its concepts. With 417 well-chosen problems the book provides ample opportunity to acquire such familiarity. The introduction (pp. 11–41) lists the facts and formulae which are supposed to be known. The problems are then divided into five chapters (pp. 42–173). A few typical problems are solved in the text, pp. 174–300 contain the solutions or hints for the solution for the remaining problems. There follows a list of references containing 63 items. 151 good figures illustrate the text.

The first chapter consists of 52 problems requiring proofs for theorems of absolute and hyperbolic geometry. Chapter II, problems 53–152, deals mostly with hyperbolic trigonometry. Chapter III, problems 153–247, treats the basic construction problems. It begins with an interesting historical introduction. Chapter IV, problems 248–342, discusses more advanced methods of construction based on a general theory of geometric loci in the hyperbolic plane and on various algebraic methods. The last chapter, problems 343–417, is concerned with problems of constructability. The tools permitted in the constructions are ruler and compass, sometimes also instruments drawing limit circles and equidistant curves. The author has proved, however, and the proof is reproduced in §31, that any construction feasible

with the four instruments can also be accomplished by ruler and compass alone. *H. Busemann* (Auckland).

Lauffer, R. Die Schwerpunkte eines nichteuklidischen Dreieckes. Jber. Deutsch. Math. Verein. **55**, Abt. 1, 119–124 (1952).

Continuing his investigation of a non-Euclidean triangle, the author, who defined the centroid of the second kind in an earlier paper [same Jber. **55**, 70–76 (1952); these Rev. **13**, 862], now defines the centroids of the third kind to be the four points whose trilinear polars coincide with their absolute polars. Taking the absolute polarity in its general form $X_i=\sum a_{ik}x_k$, so that the triangle under consideration can be the triangle of reference, one finds that the absolute polar $[X_1, X_2, X_3]$ of the point (x_1, x_2, x_3) coincides with the trilinear polar $[1/x_1, 1/x_2, 1/x_3]$ if

$$\sum a_{1k}x_1x_k=\sum a_{2k}x_2x_k=\sum a_{3k}x_3x_k.$$

Thus the centroids of the third kind are the four common points of the two conics

$$\sum a_{1k}x_1x_k=\sum a_{2k}x_2x_k,\quad \sum a_{2k}x_2x_k=\sum a_{3k}x_3x_k.$$

Each of these conic loci is apolar to the absolute conic-envelope $\sum\sum A_{ik}X_iX_k=0$, since

$$\sum a_{1k}A_{1k}=\sum a_{2k}A_{2k}=\sum a_{3k}A_{3k}.$$

Hence the quadrangle formed by their common points is the kind whose opposite sides are conjugate (i.e., perpendicular); in other words, the four centroids of the third kind are an orthocentric set, like the in- and ex-centers.

H. S. M. Coxeter (Toronto, Ont.).

Baer, Reinhold. The group of motions of a two dimensional elliptic geometry. Compositio Math. **9**, 241–288 (1951).

The elements of an arbitrary group G are considered both as the points and as the hyperplanes of a space $D(G)$. The point p lies in the hyperplane h if ph is an involution. The system $D(G)$ is self dual. The point p depends on the point set S, if p lies in every hyperplane containing all points in S. The totality of points dependent on S is the subspace spanned by S. An S-group is a group G for which $D(G)$ is a projective space, that is, satisfies the 4 conditions: 1) If the point p depends on the point q then $p=q$; 2) for a given pair of distinct points there is a third point dependent on the pair; 3) a point dependent on S depends on a finite subset of S; 4) the totality of subspaces of $D(G)$ is a complete, complemented, modular lattice. G can then be represented in a natural way as a group Λ of linear transformations T with the two properties: if $T\neq 1$ then the space of fixed elements of T has rank 1; for every subspace Q of rank 1 there is an involution in Λ with Q as space of fixed elements. The main result is that $D(G)$ is a projective space of dimension greater than 1 if and only if G is isomorphic to the motion group of a plane elliptic geometry. Four different characterizations of S-groups are obtained by interpreting the motion group of an elliptic plane either as an abstract group, or as a group of linear transformations, or as a group of projectivities of the plane on itself, or by considering the geometric properties of $D(G)$.

H. Busemann (Auckland).

Favard, J. Sur les axiomes de la géométrie. Collectanea Math. **4**, 55–69 (1951).

Observing that the cyclic axioms of order and the projective form of Pasch's axiom suffice to construct the projective plane, the objective of the present paper is to establish an analogous result for elementary geometry. The novelty of the treatment lies in postulating the following modification of Pasch's axiom: If a, b, c are three non-collinear points, let c' be a point of the line ab which is not between a and b, and let b' be a point of the line ac that is between a and c. Then the line $b'c'$ cuts the line bc in a point a' between b and c. The plane and three-space are then constructed in the usual manner. *L. M. Blumenthal.*

Ellis, David. Notes on abstract distance geometry. I. The algebraic description of ground spaces. Tôhoku Math. J. (2) **3**, 270–272 (1951).

The definitions of ground space, metroid, generalized semimetric, basality, quasigroup, loop are given in Ellis, Publ. Math. Debrecen **2**, 1–25 (1951); Monatsh. Math. **55**, 185–187 (1951); these Rev. **13**, 270, 377; Albert, Trans. Amer. Math. Soc. **54**, 507–519 (1943); **55**, 401–419 (1944); these Rev. **5**, 229; **6**, 42. A generalized semimetric ground space G is a "normal ground space" if it satisfies $d(x, 0)=x$ for all $x\epsilon G$. G has the property of "triangular fixity" if $d(x, d(x, y))=y$ for all $x, y\epsilon G$. A groupoid with unit element 0 is "nilpotent" (of index 2) if $xx=0$ for all $x\epsilon G$. Seven theorems are enunciated, without their easy proofs, correlating distance-theoretic properties of a ground space with algebraic properties of its metroid, e.g. Th. 7: A ground space whose metroid is a nilpotent Abelian group is normal, basal and triangularly fixed. The results are illustrated by two examples: $\mathfrak{R}^*$, the non-negative ray with Euclidean distance, and $\mathfrak{B}$, any autometrized Boolean algebra [Canadian J. Math. **3**, 87–93 (1951); these Rev. **13**, 377]. It follows from Th. 7 that $\mathfrak{B}$ is normal, basal and triangularly fixed, properties established in a different way in the paper just quoted. In the last paragraph problems are proposed.

C. Y. Pauc (Cape Town).

Rossier, Paul. Expérience et raisonnement en géométrie. Rev. Gén. Sci. Pures Appl. **59**, 133–147 (1952).

Convex Domains, Extremal Problems, Integral Geometry

Rado, R. Theorems on the intersection of convex sets of points. J. London Math. Soc. **27**, 320–328 (1952).

For a set $X\subset E^n$ let $|X|$, conv X, and lm X denote respectively the cardinal number of X, the convex hull of X, and the smallest linear manifold containing X. For integers $n\geqq 1$ and $r\geqq 2$, let $f(n, r)$ be the least integer N such that each $X\subset E^n$ for which $|X|=N$ can be divided into r mutually exclusive sets S_i such that $\bigcap_1^r$ conv S_i is non-empty. The author first shows that $f(n, 2)=n+2$ and that this fact can be used to prove Helly's theorem on the intersection of convex sets. (In an axiomatic treatment of Helly's theorem, this was noted also by F. W. Levi [J. Indian Math. Soc. **15**, 65–76 (1951); these Rev. **13**, 271].) He proves, more generally, that always $f(n, r)\leqq(r-2)2^n+n+2$. Equality holds if $n=1$ or $r=2$, but $f(2, 3)=7$ so the problem of completely determining $f(n, r)$ remains open. Another of the author's results is as follows: Suppose $S_1, \cdots, S_r$, are subsets of E^n such that $\bigcap_1^r$ conv S_i is non-empty. Then there are sets $T_i\subset S_i$ such that each set conv S_i is a simplex,

$$\sum_1^r|T_i|\leqq nr+r-n,\quad \bigcap_1^r \text{conv } T_i=\bigcap_1^r \text{lm } T_i,$$

and this set contains exactly one point. *V. L. Klee, Jr.*

Bambah, R. P., and Rogers, C. A. Covering the plane with convex sets. J. London Math. Soc. 27, 304–314 (1952).

The first part of this paper deals with a result of the reviewer [Acta Sci. Math. Szeged 12, Pars A, 62–67 (1950); these Rev. 12, 352] which may be stated as follows: If a convex hexagon with area H is covered by n congruent convex discs the boundaries of which intersect one another in at most two points, then $n \geqq H/h$, where h denotes the area of the largest hexagon inscribed in a disc. The proof of this theorem rests on the fact that the discs can be contracted to non-overlapping convex polygons which together cover the hexagon. The authors give a detailed and exact proof of this fact. In the second part two theorems are stated without proofs concerning the problem of covering a convex set with congruent and homothetic convex discs. The theorems are analogous to those of Rogers [Acta Math. 86, 309–321 (1951); these Rev. 13, 768] relating to spacing problems. The third part contains various complements to theorems of the reviewer and Fáry [Math. Ann. 122, 205–220 (1950); these Rev. 12, 526] on covering the plane with congruent and homothetic convex discs.

L. Fejes Tóth (Veszprém).

Hadwiger, H. Translationsinvariante, additive und stetige Eibereichfunktionale. Publ. Math. Debrecen 2, 81–94 (1951).

Let $\varphi(A)$ be a function defined for all bounded closed convex sets A of the plane, invariant under translations of A, and additive. The latter property means: if the chord $A\cap B$ decomposes $A\cup B$ (supposed convex) into A and B, then $\varphi(A\cup B)=\varphi(A)+\varphi(B)+\varphi(A\cap B)$. The function $\varphi(A)$ is continuous if $A\to A'$ in the usual sense of the theory of convex bodies implies $\varphi(A)\to\varphi(A')$. If $\beta(\varphi)$ is a continuous function with period 2π, denote by $L_\beta(A)$ the integral $\int\beta(\varphi_s)ds$, where φ_s is the angle which the tangent of the curve A' bounding A forms with a fixed direction at the point corresponding to the value s of the arc length. The main theorem is: $\varphi(A)$ is a continuous and additive function, invariant under translations of A if and only if constants α, γ and a function $\beta(\varphi)$, all independent of A, exist such that $\varphi(A)=\alpha+L_\beta(A)+\gamma F(A)$, where $F(A)$ is the area of A. The degree of arbitrariness of $\beta(\varphi)$ is discussed. The known form $\varphi(A)=\alpha+\beta L(A)+\gamma F(A)$, where $L(A)$ is the length of A', for φ which are invariant under all motions of A is easily derived from the above result. *H. Busemann.*

Zedek, Mishael. On the Jordan property and starshapedness of generalized lemniscates. Riveon Lematematika 5 (1951–1952), 62–73 (1952). (Hebrew. English summary)

Fekete's generalized lemniscate is the locus of points Q in the Euclidean plane which satisfy the equation $\sum_{i=1}^{n} g(\overline{QP_i})=ng(R)$ where the P_i are arbitrary fixed points, $g(r)$ is strictly monotonic decreasing with $\lim_{r\to0+}g(r)=\infty$ and R is a positive number. The author proves: (1) If d is the diameter of the set $\{P_i\}$ then the lemniscate is a Jordan curve for $R\geqq d$, while for $R<d$ there exists a $g(r)$ such that the lemniscate is not a Jordan curve. (2) If $D=\max_{ij}(\overline{P_jM_i}+\overline{M_iO})$ where M_i is the midpoint of the segment OP_i, then the lemniscate is starshaped with respect to O for $R\geqq D$, while for $R<D$ there exists a $g(r)$ such that the lemniscate is not starshaped with respect to O.

E. G. Straus (Los Angeles, Calif.).

Aleksandrov, A. D. On surfaces represented as the difference of convex functions. Izvestiya Akad. Nauk Kazah. SSR. 60, Ser. Mat. Meh. 3, 3–20 (1949). (Russian. Kazak summary)

The paper gives detailed proofs for the results which were listed earlier without proof [Doklady Akad. Nauk SSSR 72, 613–616 (1950); these Rev. 12, 353]. It is shown that all functions $f(x, y)$ whose first derivatives exist and satisfy a Lipschitz condition may be represented as difference of convex functions, and also all functions which represent polyhedra. *H. Busemann* (Auckland).

Ohmann, D. Eine Minkowskische Ungleichung für beliebige Mengen und ihre Anwendung auf Extremalprobleme. Math. Z. 55, 299–307 (1952).

The author employs methods similar to those of his preceding paper [Math. Ann. 124, 265–276 (1952); these Rev. 13, 864] to generalize the Minkowski mixed volume inequality $V(A, K\cdots K)^n\geqq V(A)V(K)^{n-1}$. He supposes that the convex body K has central symmetry and he replaces A by an arbitrary set of points and V by suitable inner or outer mixed volumes. *L. C. Young* (Madison, Wis.).

Fejes Tóth, L. Elementarer Beweis einer isoperimetrischen Ungleichung. Acta Math. Acad. Sci. Hungar. 1, 273–276 (1950). (German. Russian summary)

Given a simply connected polygon in the euclidean plane. Notations: F=area, L=length, r=radius of in-circle, R=radius of circum-circle. The author gives a simple elementary proof of the following inequalities:

$$L^2-4\pi F\geqq(L-2\pi r)^2,\quad L^2-4\pi F\geqq(2\pi R-L)^2$$

[cf. Besicovitch, Quart. J. Math., Oxford Ser. 20, 84–94 (1949); these Rev. 11, 51]. Both of them are contained in

$$(1)\qquad 4L\rho\geqq2(F+L\rho+\pi\rho^2)\quad[r\leqq\rho\leqq R].$$

The proof of (1) is based on geometrical interpretations of the quantities $4L\rho$ and $F+L\rho+\pi\rho^2$. The interpretation of the former is a special case of a theorem by Poincaré and that of the latter one is contained in Blaschke's "kinematische Hauptformel" [cf. Blaschke, Vorlesungen über Integralgeometrie, vol. I, 2nd ed., pp. 24 and 37, Teubner, Leipzig-Berlin, 1936]. *P. Scherk* (Los Angeles, Calif.).

Santaló, L. A. Integral geometry in Hermitian spaces. Amer. J. Math. 74, 423–434 (1952).

In einem n-dimensionalen elliptisch-hermitischen Raum werden für linearen Untermannigfaltigkeiten L_r Dichten bestimmt. Ist L_r^0 eine r-dimensionale lineare Untermannigfaltigkeit die bei den Transformationen der Untergruppe Γ_r der unitären Gruppe U des Raumes invariant bleibt, dann ist die Dichte dL_r von L_r nichts anderes als das Raumelement des durch U/Γ_r bestimmten homogenen Raumes. Die analytischen Ausdrücke für die Dichte bekommt Verf. unter Verwendung der Cartanschen Methode des beweglichen Bezugssystems, nach einem von ihm [Ann. of Math. 51, 739–755 (1950); diese Rev. 11, 681] und S. S. Chern [ibid. 43, 178–189 (1942); diese Rev. 3, 253] herrührenden Verfahrens. Dazu wird ein n-Simplex gewählt, dessen Eckpunkte in Bezug auf das absolute Gebilde des Raumes selbstkonjugiert ist. Die Relativkomponenten des beweglichen Bezugssystems mögen durch die Pfaffschen Formen ω_{kj} $(k, j=0, 1, \cdots, n)$ bestimmt werden und ω^*_{kj} bezeichne die zu ihnen konjugierten Formen. Im Falle einer Unter-

gruppe Γ_r müssen $\omega_{jk}=0$ sein für $0\leqq j\leqq r$, $r+1\leqq k\leqq n$. Die Dichte dL_r ist dann, bis auf einen konstanten Faktor, durch den Absolutbetrag des äusseren Produktes $dL_r=\prod\omega_{is}\omega^*{}_{is}$, $0\leqq i\leqq r$, $r+1\leqq s\leqq n$ der $2(r+1)(n-r)$ nicht verschwindenden Pfaffschen Formen bestimmt. Die kinematische Dichte d.h. das Raumelement für die Gesamtgruppe U ist, abgesehen von einen konstanten Faktor, durch $du=\prod\omega_{jk}\omega^*{}_{jk}\prod\omega_{hh}$, $j<k$, $0\leqq j$, k, $h\leqq n$ bestimmt. Ist C_p eine p-dimensionale analytische Mannigfaltigkeit $x^i=x^i(t_1,\cdots,t_p)$, dann existiert für dieselbe ein invariantes Integral J_p. Dasselbe kann folgendermassen definiert werden. Sind die projektiven Koordinaten x^i so normiert, dass $x^ix^{*i}=1$ ist, und setzt man $\Omega^p=\sum dx^{i_1}\cdot dx^{*i_1}\cdots dx^{i_p}dx^{*i_p}$ wobei die Summation über die einzelnen äusseren Formen sich auf sämtliche Kombinationen $i_1\cdots i_p$ von 1 bis n erstreckt, dann ist $J_p(C_p)=p!/(2\pi i)^p\int_{C_p}\Omega^p$. Für eine algebraische Mannigfaltigkeit C_p ist $J_p(C_p)$ die Ordnung derselben. Sind C_r und C_h zwei Mannigfaltigkeiten für die $r+h\geqq n$ ist und wird C_r durch die Transformation $u\subset U$ in uC_r überführt, dann findet Verf.

$$\int_U J_{h+r-n}(C_h\cap uC_r)du=J_h(C_h)J_r(C_r).$$

Dabei ist das Raumelement du so genormt, dass das Gesamtmass von U gleich eins ist. Falls an Stelle von C_r eine lineare Untermannigfaltigkeit L_r betrachtet wird, so ergibt sich

$$\int_{U/\Gamma_r} J_{h+r-n}(C_h\cap L_r)dL_r=J_h(C_h);$$

wieder ist dL_r so genormt, dass der Gesamtinhalt von U/Γ_r gleich eins ist. *O. Varga* (Debrecen).

Blaschke, Wilhelm. Zur Integralgeometrie. Rend. Circ. Mat. Palermo (2) 1, 108–110 (1952).

The author suggests a method to prove the kinematic formula in integral geometry:

$$\int C(G_0G_1)dG_1=8\pi^2(V_0C_1+F_0M_1+M_0F_1+C_0V_1).$$

Here G_0, G_1 are two domains in ordinary Euclidean space, of which G_0 is fixed and G_1 is moving; dG_1 is the kinematic measure of G_1; C_i, M_i, F_i, V_i are respectively the total curvature, integral of mean curvature, area of boundary surface, and volume of G_i, $i=0, 1$. The method is to start from the formula

$$\int V(G_0G_1)dG_1=8\pi^2V_0V_1,$$

pass to domains parallel to G_i at a distance h, and let $h\to 0$. *S. Chern* (Chicago, Ill.).

Varga, Ottó. The application of integral geometry in geometrical optics. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 1, 192–201 (1951). (Hungarian)

An expository lecture on integral geometry; geometrical optics is mentioned only briefly as one place where it is relevant to assign measure to sets of lines. *P. R. Halmos* (Chicago, Ill.).

Algebraic Geometry

✓★**Hodge, W. V. D., and Pedoe, D. Methods of algebraic geometry. Vol. II. Book III: General theory of algebraic varieties in projective space. Book IV: Quadrics and Grassmann varieties.** Cambridge, at the University Press, 1952. x+394 pp. $7.50.

While in the first volume of this treatise [these Rev. 10, 396] the authors dealt with algebraic preliminaries and projective spaces, in this second volume they initiate a systematic exposition of algebraic geometry proper. The first three chapters (Book III, chapters X–XII) develop respectively the general concept of an algebraic variety, the theory of algebraic correspondences and intersection theory, while the remaining two chapters (Book IV, chapters XIII and XIV) deal respectively with the theory of quadrics and Grassmann varieties and serve also the purpose of illustrating the general methods developed in the preceding chapters. It is stated in the introduction that the arithmetic theory of varieties and the foundations of birational geometry will be treated in a third volume. The authors restrict themselves to ground fields of characteristic zero but allow ground fields which are not algebraically closed (except in the last two chapters), in conformity with the by now well established fact that even classical algebraic geometry can profit from a relative theory of varieties over arbitrary ground fields (of characteristic zero). The exposition is based on algebraic methods of recent vintage (van der Waerden and to a lesser extent A. Weil and the reviewer) and is flawless as to precision and rigor. The combination of strictly algebraic methods and strictly geometric aims and motivation makes this volume, within the limits of the topics treated, an excellent bridge between the old and the new in algebraic geometry.

Chapter X presents the fundamental concepts associated with an algebraic variety (irreducibility relative and absolute, dimension, general points, etc.) and also develops at great length the properties of the associated form of a variety (Chow-van der Waerden; for historical reasons the authors prefer to call this the Cayley form). The effective use of the associated form is a basic feature not only of this chapter but also of the whole book III. The chapter ends with a study of absolutely irreducible varieties, including the reviewer's characterization of such varieties and his proof of the absolute irreducibility of their general hyperplane sections.

The general theory of algebraic correspondences as developed in the next chapter (XI) is based mainly on the work of van der Waerden (and on elimination theory). The product variety of two projective spaces (Segre varieties) is introduced, but in actual work with correspondences the authors prefer to use multiple projective spaces. The usual dimension-theoretic properties of correspondences are given, including the principle of counting constants and the much more informative theorem on p. 115 (Theorem I). Following A. Weil, the notion of a cycle is introduced and algebraic correspondences are accordingly re-examined from this new point of view. In connection with what the authors call "normal problems" (of which the principle of Plücker-Clebsch furnishes a first example) they appeal for the first time to arithmetic properties of normal varieties in order to prove the theorem of "uniqueness of specialization" and to derive the "criterion for unit multiplicity", thus preparing the ground for the intersection theory developed in the next chapter. Following van der Waerden, the authors make use of the projective group of transformations to develop first a global intersection theory in projective spaces, but they

extend considerably this work so as to include a relative (global) intersection theory on any non-singular variety. In this chapter (XII) the authors also develop the theory of equivalence of cycles on a variety and prove the existence of a base for Segre varieties. The local intersection theory (which presents other serious difficulties) is not treated in this volume. However, it is quite true, as the authors observe, that a global intersection theory is sufficient for most applications to problems of classical algebraic geometry.

The material covered in the next chapter on quadrics is standard but extensive (linear spaces on a quadric, subvarieties of a quadric, the elementary divisors of orthogonal matrices, pairs of quadrics, to mention some of the topics treated) and is characterized by a very careful and rigorous treatment for which the general methods developed in the preceding chapters provide the necessary tools. This is also true of the last chapter on Grassmann varieties which contains a good deal of the original work of Severi, Hodge and Ehresmann. Among the topics treated in this chapter we note the following: the Schubert calculus; the basis theorem on Grassmann varieties; intersection formulae and applications to enumerative geometry; proof of Severi's theorem that the subvarieties of maximum dimension on a Grassmann variety are complete intersections. *O. Zariski.*

Wolkowitsch, David. Pentagones et pentaèdres conjugués à une quadrique. C. R. Acad. Sci. Paris **233**, 1415–1416 (1951).

If the tangential equations $(A_k)=0$ of r points satisfy an identity of the form $\sum_{k=1}^{r} m_k(A_k)^2=0$, any quadric through $r-1$ of them passes through the rth also [see p. 224 of the book reviewed above]. In particular, when $r=8$ we have eight associated points, which may be regarded (in 35 ways) as two tetrahedra both self-polar for the same quadric [H. F. Baker, Principles of geometry, vol. III, Cambridge Univ. Press, 1923, pp. 148, 154]. The author points out that any ten points on a quadric satisfy such an identity and consequently may be regarded (in 126 ways) as two complete pentagons both self-conjugate for the same quadric. (A self-conjugate pentagon consists of five points such that the pole of the plane of any three is collinear with the other two [Baker, ibid., p. 50].)

By counting parameters, he infers that any three quadrics have a common self-conjugate pentagon. The fallacy in this argument was pointed out by G. Salmon [Analytic geometry of three dimensions, vol. 1, 6th ed., Longmans Green, London, 1914, pp. 137, 242, footnotes].

H. S. M. Coxeter (Toronto, Ont.).

✓***Locher-Ernst, Louis. Einführung in die freie Geometrie ebener Kurven.** Verlag Birkhäuser, Basel, 1952. 88 pp. 12.50 Swiss francs.

This book is a simple and readable account of the behaviour of curves in the real projective plane, based on essentially topological notions and the idea of continuity; no algebra at all is employed, though the net result of a careful reading is an insight into the properties of algebraic curves and what they look like, which the ordinary algebraic treatments hardly give. The axiomatic material is of the simplest, and the continuity properties involved are all expressed in terms of the continuous motion of a point along a line, and (dually) the continuous rotation of a line about a point. An elementary arc is the self-dual manifold of points X and lines x (in one-to-one correspondence) which can be continuously described, i.e. so that if O is an arbitrary fixed point and o an arbitrary fixed line the line OX moves continuously in the pencil and the intersection ox moves continuously on o; and each x is the tangent (well defined, but the definition is too long to quote) at the corresponding X. If the initial and final configurations of the pair (X, x) coincide, the arc is an elementary curve. The inflexion, "Dornspitze", and "Schnabelspitze", are defined by the change of sign in the motion of ox, of OX, and of both, for general O and o. The intersections of an arc with a line, and the tangents drawn to it from a point are studied, and multiplicity simply defined from considerations of continuity; thus the order and class of an elementary curve are defined, and many important results proved, such as the classification of elementary curves of order 3 into those of class 6 with three inflexions, of class 4 with three inflexions, of class 4 with an inflexion and a double point, and of class 3 with an inflexion and a cusp. Finally, the resolution of a double point by "rounding off" a pair of opposite angles ("Ecken") and the dual treatment of double tangents, with the effect of these processes on order and class, are very lucidly studied, and numerous examples of curves of order or class 4 examined. There are nearly 200 well drawn figures, which unfortunately are all together at the end.

P. Du Val (Bristol).

Dedò, Modesto. Algebra delle trecce caratteristiche: relazioni fondamentali e loro applicazione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 227–258 (1950).

Chisini has introduced [same Rend. (2) **66**, 1141–1155 (1933)] a braid [cf. Artin, Abh. Math. Sem. Univ. Hamburg **4**, 47–72 (1926); Ann. of Math. **48**, 101–126 (1947); these Rev. **8**, 367] associated with a plane algebraic curve. This braid is by no means unique, and different methods of constructing the braid can lead to different braids. The author investigates how changes in the method of construction affect the resulting braid, introducing a few fundamental operations by the application of which he passes from one braid to an equivalent one (in the sense that, if the first braid represents a curve, the second braid could represent the same curve). The author discusses the manner in which his elementary operations combine, and then deals with various canonical forms for the braids of plane cubic curves. The results are applied to the plane quartic with three cusps and the sextic with six cusps lying on a conic. He is able to verify from the braids in these cases that the curves are branch curves of triple planes, since they satisfy Enriques' conditions of invariance [Ann. Mat. Pura Appl. (4) **1**, 185–198 (1924)]. *D. B. Scott* (London).

Masotti Biggiogero, Giuseppina. Sulle curve di diramazione dei piani quadrupli generali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **13(82)**, 396–400 (1949).

Dans un mémoire antérieur, l'auteur [mêmes Rend. (3) **11(80)**, 269–280 (1949); ces Rev. **11**, 738] a donné une caractérisation des courbes planes φ_{12} qui sont courbes de branchement d'un plan quadruple obtenu par projection d'une surface F^4 d'un point extérieur. Une telle courbe, qui a 24 rebroussements constituant un groupe K_{24} et 12 points doubles définissant un groupe N_{24} de places, doit vérifier les trois conditions nécessaires et suffisantes suivantes. 1) Les sections rectilignes R appartiennent à une g_{12}^{3+s} $(s\geqq 0)$, c'est à dire que φ_{12} est normale dans un espace à au moins trois dimensions. 2) K_{24} et N_{24} appartiennent à la série $|2R|$. 3) La g^1_{24} définie par K_{24} et N_{24} contient un groupe découpé effectivement sur φ_{12} par une conique. La question restait à résoudre de savoir si ces trois conditions sont indépendantes.

Dans le présent mémoire, l'auteur montre que la condition 2) et l'irréductibilité de φ_{12} suffisent à entraîner la condition 3). D'où résulte que pour les courbes φ_{12} irréductibles, les conditions 1) et 2) sont à elles seules nécessaires et suffisantes.

L. Gauthier (Nancy).

Masotti Biggiogero, Giuseppina. Sulla caratterizzazione delle curve di diramazione dei piani quadrupli. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 12(81), 95–112 (1948).

Étudiant les courbes de branchement des plans quadruples généraux, l'auteur a caractérisé les courbes φ_{12} qui sont courbes de branchement pour un plan quadruple obtenu par projection d'une surface F^4 d'un point extérieur [mêmes Rend. (3) 11(80), 269–280 (1947); ces Rev. 11, 738; complété par le mémoire analysé ci-dessus]. Dans le mémoire actuel, l'auteur donne la condition nécessaire et suffisante pour qu'une courbe φ d'ordre $6(n-2)$ ayant $6(n-2)^2$ rebroussements et $4(n-1)(n-3)$ points doubles soit courbe de branchement d'un plan quadruple obtenu par projection d'une surface F^n d'un point multiple d'ordre $n-4$. Il faut et il suffit qu'il existe sur φ un groupe T de $(n-3)(n-4)$ points simples situés sur une C^{n-4} vérifiant les trois propriétés suivantes. a) La série $|R+T|$, où R est un groupe de points alignés de φ, est sans point fixe, de dimension $\geqq 3$. b) Le groupe K des rebroussements est équivalent à $(n-2)R$, et le groupe $2N$ des places situées aux points doubles est équivalent à $2(n-3)R-4T$. c) Il existe une courbe d'ordre $2(n-3)$ qui découpe sur φ un groupe appartenant à la série linéaire définie par $2N+4T$ et $K+R'$, où R' est le groupe découpé par C^{n-4}.

L'auteur construit sa démonstration sur le cas $n=5$ qui présente tous les caractères essentiels d'une démonstration générale: on commence par étudier diverses courbes projectivement covariantes à φ. Ceci permet ensuite de déterminer l'équation d'une courbe φ ayant les propriétés a), b), c) et enfin d'obtenir explicitement l'équation de la surface F^5 qui l'admet pour courbe de branchement. Il est possible que la condition c) soit conséquence des conditions a) et b), comme cela a lieu pour $n=4$, mais ce fait n'est nullement évident.

L. Gauthier (Nancy).

Masotti Biggiogero, Giuseppina. Sopra un teorema di Liouville. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 735–752 (1950).

The paper begins with a new proof of the theorem [J. Liouville, J. Math. Pures Appl. 6, 345–411 (1841)] that the sum of the cotangents of the angles of intersection between two plane algebraic curves, none of whose intersections are at infinity, depends only on the points at infinity of the two curves. (The proof is by varying one of the curves and shewing that the sum in question remains finite in all the cases where it is not obviously so, and hence is constant.) A number of simple corollaries and special cases are deduced, of which the most interesting are:

For two conics, the sum in question is zero if and only if either the axes of one are parallel to those of the other, or the asymptotic directions of one are harmonic to those of the other.

If one of the curves, of order m, has all its points at infinity in the circular points (h in one and $m-h$ in the other), the sum of the cotangents is $(2h-m)i$, independently of the second curve; and this is the only case in which the sum is independent of one of the curves. (As a by-product of the proof here the author obtains the curious identity

$$\begin{vmatrix} \lambda & -1 & 0 & \cdots 0 & 0 & 0 \\ m & \lambda & -2 & \cdots 0 & 0 & 0 \\ 0 & m-1 & \lambda & \cdots 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots \lambda & -(m-1) & 0 \\ 0 & 0 & 0 & \cdots 2 & \lambda & -m \\ 0 & 0 & 0 & \cdots 0 & 1 & \lambda \end{vmatrix} \begin{aligned} &\equiv \lambda(\lambda^2+2^2)(\lambda^2+4^2)\cdots \\ &\quad \times(\lambda^2+m^2) \text{ if } m \text{ is even} \\ &\equiv (\lambda^2+1^2)(\lambda^2+3^2)\cdots \\ &\quad \times(\lambda^2+m^2) \text{ if } m \text{ is odd.)} \end{aligned}$$

If one of the curves of order m is given, the other curve is to be of order n, and the sum of the cotangents is to have an assigned value; the possible configurations of the points at infinity of the second curve form an algebraic manifold of order m and dimension $n-1$, i.e. all but one of the n points can be chosen at will and there are then in general m possible positions for the nth point.

P. Du Val (Bristol).

Marchionna, Ermanno. Condizioni caratteristiche perchè una curva sia di diramazione per un piano multiplo. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 655–664 (1950).

Marchionna, Ermanno. Una nuova caratterizzazione delle curve di diramazione dei piani multipli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 11, 170–177 (1951).

In these two papers the following theorem is proved, completely in the second; in the first the statement is weaker and the proof limited to a numerically special case. The necessary and sufficient conditions for a plane n-ic curve φ_n, with δ nodes and k cusps, to be the branch curve of a $(\mu-h)$-ple plane $(\mu-h\geqq 3)$, which is the projection of a μ-ic surface A from an h-ple point O of itself (A being general except for this h-ple point), are the following numerical relations:

$$n=\mu(\mu-1)-h(h+1),$$
$$\delta=\tfrac{1}{2}n[(\mu-2)(\mu-3)-h(h+1)]-2h(h+1)(\mu-h-3),$$
$$k=\mu(\mu-1)(\mu-2)-h(h+1)(h+2)$$

together with the relation of equivalence on φ_n:

$$2K+N'\equiv[(\mu-1)(\mu-2)-h(h+1)]R-(2\mu-2h-4)T,$$

where K is the set of cusps, N' that of nodes (counted on both branches), R a collinear set, and T a set of $h(h+1)$ simple points, such that $R+T$ is equivalent to some sets having no point in common with T, i.e. the series $|R+T|$ is of freedom $\geqq 3$ and has no fixed points. T is in fact the image of the $h(h+1)$-ple point O on the curve φ^*, intersection of A with the polar of O, of which of course φ_n is the projection. K and N' are given separately by

$$K\equiv(\mu-2)R+(\mu-h-4)T,$$
$$N'\equiv[(\mu-2)(\mu-3)-h(h+1)]R-4(\mu-h-3)T.$$

P. Du Val (Bristol).

Marchionna, Ermanno. Estensione di un teorema di Halphen (relativo a curve gobbe intersezioni complete). Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 137–158 (1950).

The following theorems are proved:

I. If Γ^* is the curve of intersection of two surfaces of orders μ, ν, having multiplicities h, k in a point O^* (their tangent cones there having no common constituent), the plane projection Γ of Γ^* from a general point is $\mu\nu$-ic with an hk-ple point in the projection O of O^*, and its further double points constitute its complete residual intersection

with a curve of order $(\mu-1)(\nu-1)$ passing through O with multiplicity $(h-1)(k-1)$.

II. Conversely, if a space curve Γ^* of order $\mu\nu$ ($\mu\geqq\nu$) with an hk-ple point O^* projects from a general point into a plane curve satisfying the above conditions, then Γ^* is the complete intersection of two surfaces $\bar{A}_\mu$, B_ν (of orders μ, ν) and hence the base of a linear system Σ of surfaces

$$A_\mu=\bar{A}_\mu+B_\nu Q_{\mu-\nu}$$

(where $Q_{\mu-\nu}$ is a general surface of order $\mu-\nu$). The multiplicity in O^* of the general A_μ is the lesser of h, k, and that of B_ν may be anything between h and k (inclusively). If $h\neq k$ the surfaces of Σ have at least the same tangent cone in O^*. *P. Du Val* (Bristol).

Marchionna, Ermano. Sull'intersezione di due superficie aventi un punto multiplo in commune. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 290–296 (1950).

Continuing the investigation in the paper reviewed above, the author shews:

III. A necessary and sufficient condition for Γ^* (of order $\mu\nu$ and with an hk-ple point O^*) to project into a plane curve satisfying the conditions of I is that its canonical series be traced on it by surfaces of order $\mu+\nu-4$ with an $(h+k-2)$-ple point in O^*. Hence:

IV. If on Γ^*, of order $\mu\nu$ and with an hk-ple point O^*, the canonical series is traced by surfaces of order $\mu+\nu-4$ with an $(h+k-2)$-ple point at O^*, the whole conclusion of II follows.

It is then pointed out that the space curves of given order $\mu\nu$ with a point of given multiplicity hk which are complete intersections are not always exhausted by the two obvious systems in which the μ-ic surface has an h-ple and the ν-ic a k-ple point, and that in which the roles of h, k are interchanged. There may be also a family in which the multiplicities of the two surfaces are lower but their tangent cones have a common constituent. *P. Du Val.*

Bagchi, Haridas, and Mukherji, Biswarup. Note on a circular cubic with a real coincidence point at infinity. Bull. Calcutta Math. Soc. 43, 101–108 (1951).

Bagchi, Haridas, and Mukherji, Biswarup. Note on a circular cubic, having one or more sextactic points at infinity. I. Rend. Sem. Mat. Univ. Padova 20, 365–380 (1951).

The equation of a circular cubic is taken in the form

$$(x-\lambda)(x^2+y^2)+ax+by+c=0$$

so that the origin is the "double focus", intersection of the imaginary (isotropic) asymptotes, and $x=\lambda$ is the real asymptote. The conditions on (λ, a, b, c) in order that (i) the real point at infinity K, (ii) the double focus 0, (iii) either or both of the circular points at infinity I, J, may be sextactic, and in each case the equation of the corresponding osculating conic, are found. It is shewn also that if 0 is sextactic, so is K; if K is sextactic and 0 lies on the curve, 0 is also sextactic; if any two of I, J, K are sextactic, so is the third; and if these are sextactic and 0 lies on the curve, the curve is central and 0 is its centre (which is of course an inflexion). *P. Du Val* (Bristol).

Bagchi, Haridas, and Mukherji, Biswarup. Note on the circular cubic and bi-circular quartics with four assigned cyclic points. II. Rend. Sem. Mat. Univ. Padova 20, 381–388 (1951).

The authors claim that some of the following results are new, but it is not made clear which. 1) Given any four concyclic points there is a unique circular cubic having these as cognate cyclic points. (By a cyclic point seems to be intended a point with stationary curvature, i.e. whose osculating curvature has four point contact; the circular cubic or quartic has 16 such points lying by fours on the four circles of inversion, and those on one circle are called cognate.) 2) The circular cubic is also uniquely determined if one of the four circles of inversion and the corresponding focal parabola are given (by the focal parabola seems to be meant the locus of centres of a system of bitangent circles). 3) Given any four concyclic points α, β, γ, δ the bicircular quartics having these as cognate cyclic points form a pencil; all have the same four circles of inversion; the locus of the remaining cyclic points consists of the three circles of inversion, other than that on which α, β, γ, δ lie; the locus of the proper foci consists of all four circles; the locus of the "double" foci is the unique circular cubic, passing through the centre of the circle $\alpha\beta\gamma\delta$ and through the six intersections by pairs of the tangents to this circle at α, β, γ, δ; and the eight bitangents of each curve of the pencil meet by pairs in the centres of the four circles. 4) Given any three non collinear points, the circular cubics having these as cognate cyclic points form a pencil, the locus of whose "double" focus is the circumcircle of the triangle formed by the tangents at the three points to the circle through them. *P. Du Val* (Bristol).

Atiyah, M. F. A note on the tangents of a twisted cubic. Proc. Cambridge Philos. Soc. 48, 204–205 (1952).

The author studies briefly the relation between a twisted cubic in S_3 and the Grassmannians C_4 of its tangents and V of its chords. The main results are that the section of Ω (Grassmannian of all the lines of S_3) by the ambient Σ of C_4 is the invariant quadric I of C_4 (envelope of primes meeting C_4 in equianharmonic sets) and hence that the necessary and sufficient condition for four tangents of the cubic to have only one common transversal is that their points of contact are equianharmonic; and that the surface U, locus of intersections of pairs of osculating planes of C_4, is the projection of V onto Σ from the pole of Σ with respect to Ω. *P. Du Val* (Bristol).

Bonera, Piero. Un problema sulle quintiche gobbe razionali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 475–480 (1950).

The author derives the necessary and sufficient invariant condition that a rational space quintic with four points of hyperosculation (points at each of which the osculating plane has five-point contact) have at least one node. He then shows that if this quintic has one node, it must have another, that is, precisely two. *T. R. Hollcroft.*

Bonera, Piero. Sestiche gobbe razionali dotate di quattro punti di iperosculazione e di nodi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 753–762 (1950).

The author derives the necessary and sufficient invariant condition that a rational space sextic with four distinct points of hyperosculation (points at each of which the osculating plane has six-point contact) have at least one node. He then finds that a rational sextic with four points

of hyperosculation satisfying this condition has four nodes, two nodes, or one node, according as the value of the anharmonic ratio of the four points of hyperosculation is -1, -2, or -3, respectively. *T. R. Hollcroft.*

Roth, Leonard. Sulle V_3 algebriche generate da congruenze di curve. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 66–70 (1952).

Three-dimensional varieties V which carry a linear congruence Γ of curves γ of genus 0 or 1 and a complete system $|A|$ of dimension $r\geqq 2$ of surfaces whose characteristic system is composed of the curves γ are studied. The results are applied to obtain generalizations of the criteria of Enriques [Math. Ann. 49, 1–23 (1897)] for the unirationality of a V_3. *H. T. Muhly* (Iowa City, Iowa).

Roth, Leonard. Algebraic threefolds. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 297–346 (1951).

This is an expository account of the present state of the theory of algebraic threefolds, in its various aspects. The five chapters are concerned with projective theory, invariantive theory, systems of equivalence, Riemann-Roch theorem and the theory of the base, and problems of rationality. There is an extensive bibliography listing well over a hundred papers. In view of the interest taken in the properties of threefolds at the present time, and the very incomplete state of our present knowledge, this survey is a timely undertaking. *J. A. Todd* (Cambridge, England).

Baldassarri, Mario. Sulle V_3 contenenti un sistema lineare triplamente infinito di superficie razionali. Rend. Sem. Mat. Univ. Padova 20, 135–152 (1951).

Ce mémoire généralise un travail de Morin sur la classification des V_3 à sections superficielles rationnelles [Ann. Mat. Pura Appl. (4) 18, 147–171 (1939); ces Rev. 1, 83]. L'auteur commence par étudier les surfaces F de S_3 irréductibles, admettant des sections planes rationnelles dépendant de 2 paramètres: ce sont, soit des surfaces à sections planes elliptiques (sections rationnelles par les plans tangents), soit des surfaces ayant un point multiple propre O (sections rationnelles par les plans issus de O). Ces dernières sont normales dans S_3 dans le seul cas où ce sont des monoïdes. Sinon, elles sont projections de surfaces normales F' ayant un point multiple O' d'où on peut les projeter suivant des surfaces de Veronese ou des réglées rationnelles.

Les surfaces F^n de S_3 irréductibles, admettant des sections planes elliptiques dépendant de 2 paramètres sont, soit des surfaces à sections planes de genre 2, soit des surfaces ayant un point multiple propre O. Ces dernières sont normales dans S_3 dans le seul cas où la multiplicité de O est $n-2$. Sinon elles sont projections de surfaces normales F'^n ayant un point multiple O' d'où l'on peut les projeter suivant des surfaces à sections planes elliptiques (y compris le plan double avec quartique de branchement).

L'auteur considère ensuite les V_3 de S_4 contenant un système à 3 paramètres de surfaces rationnelles. Il montre que ce système est composé avec une involution de points et que V_3 est ainsi régulière et unirationnelle. Les V_3 obtenues sont, soit les variétés à sections hyperplanes rationnelles [cf. Morin, loc. cit.], soit des variétés ayant un point multiple O. Ces dernières se rangent en trois catégories: 1) Celles qui sont normales dans S_4; ce sont des monoïdes; 2) celles qui sont projections d'une V'^n_3 ayant un point multiple O' d'où l'on peut les projeter suivant des variétés à sections superficielles rationnelles; 3) celles qui sont représentables sur un S_3 double avec une surface F^4 de branchement. Il en résulte qu'une telle V_3 est rationnelle ou référable à une $V_3{}^2$. Il en résulte également que les systèmes linéaires à 3 dimensions de surfaces rationnelles de S_3 sont: i) le système des plans de S_3; ii) les sous-systèmes de systèmes de dimension plus grande représentant les V_3 à sections superficielles rationnelles [cf. Morin, loc. cit.]. *L. Gauthier* (Nancy).

Nollet, Louis. Sur les surfaces algébriques irrégulières douées d'un faisceau de courbes de genre 2. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 873–878 (1951).

It is proved that if an algebraic surface has a linear pencil of irreducible curves of genus 2, then the surface is either regular or referable to a ruled surface of genus 1 or 2. *P. Du Val* (Bristol).

Nollet, Louis. Sur l'invariant de Zeuthen-Segre des surfaces algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 1044–1052 (1951).

En passant des courbes aux surfaces algébriques, à la série canonique d'une courbe correspond par analogie la série d'équivalence S de Severi; et comme l'ordre $2p-2$ de la série canonique sur une courbe de genre p est toujours un nombre pair, on peut se demander si l'ordre $I+4$ de la série S sur une surface irréductible F sans points multiples ni courbes exceptionnelles de première espèce peut prendre toutes les valeurs entières. L'A. se propose ici de démontrer que $I+4$ ne peut pas avoir la valeur 1. En utilisant les formules de Picard-Alexander et de Noether-Bonnesen, il trouve d'abord que, si $I+4=1$, c'est-à-dire si $I=-3$, alors la surface F a une irrégularité $q\geqq 2$, un genre $p^{(1)}\geqq 12$, n'appartient pas à la classe des surfaces réglées, ne possède pas de faisceau algébrique de courbes de genre 0 ou 1; et si elle possède un faisceau algébrique le genre de ce faisceau doit être 0 ou 1. En se servant alors d'un projection de F sur un espace à trois dimensions, et de deux intégrales simples de première espèce u_1, u_2 linéairement (et fonctionnellement) indépendants existant sur F, on arrive à une contradiction. En appendice, avec d'autres remarques, l'A. réfute un raisonnement simpliste qui semble réduire le résultat obtenu à un corollaire immédiat d'un théorème de Severi. *E. G. Togliatti* (Gênes).

Nollet, Louis. Sur un théorème de M. Severi. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 1053–1054 (1951).

Dans un mémoire de 1932 [Comment. Math. Helv. 4, 268–326 (1932)] F. Severi a démontré que sur une surface algébrique irréductible F, dont le système canonique $|K|$ n'a pas la partie variable composée avec un faisceau irrationnel de courbes, deux intégrales simples de la première espèce ayant le même groupe Jacobien coïncident à moins d'une constante multiplicative et d'un constante additive près. L'A. considère ici les cas qui ne sont pas compris dans le théorème précédent. Il trouve que le théorème est encore applicable lorsque F possède un faisceau irrationnel Γ dont le genre est égal à l'irrégularité de F; dans le cas contraire le théorème est encore applicable, sauf dans le cas où l'invariant I de Zeuthen et Segre a la valeur -4, c'est-à-dire dans le cas où F est elliptique ou hyperelliptique. *E. G. Togliatti* (Gênes).

Godeaux, Lucien. Sur les surfaces algébriques d'ordre n dont les adjointes d'ordre $n-4$ possèdent une partie fixe. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 45–56 (1951).

This paper is a contribution to the construction of canonical surfaces of genus 4, i.e., surfaces of order n in three dimensions whose canonical system is traced by planes, so that there is an adjoint surface of order $n-5$, and all those of order n consist of this and a plane. The main theorem proved is slightly stronger however than is needed for this purpose; it is as follows. To construct a surface F of order $n=2\nu$, having a double curve D of order νm and $\tau=4\epsilon$ points triple both on F and on D, such that its complete canonical system is traced by the totality of curves of order $n-m-4$ in space, the following is necessary. 1) To construct a surface Φ with τ conical nodes, which is the image of an involution of order 2 on a surface Φ^*, with τ isolated united points corresponding to the nodes of Φ, and such that the systems on Φ^* which correspond to the plane sections $|C|$ of Φ, and to the systems $|2C|, |3C|, \cdots, |(\nu-4)C|$, have respectively the same dimensions as these. Further, there must exist at least $\infty^{2\tau}$ surfaces [of order n] passing through the double points of Φ and touching it in all common points. [This ensures the existence of the curve D on Φ, having triple points at the nodes of Φ, and along which Φ is touched by a surface of order n.] 2) There must exist a surface Ψ of order $n-m$, passing simply through the nodes of Φ, and cutting Φ further in the set of nodes of some surface F_0 of order n which touches Φ along D and has triple point in the nodes of Φ. In the pencil determined by F_0 and the reducible surface $\Phi+\Psi$, there is then one which has D as double curve, and of which every adjoint surface of order $n-4$ contains Φ as part.

This theorem is applied to the construction of a canonical surface of genus 4 and order 10 (i.e., linear genus 11). The surface Φ is one of order 5 which touches a Kummer surface K along a curve of order 10, from which the existence of the surface Φ^* is deduced. D is of order 25, and has 25 triple points which are the nodes of Φ (all of which are on the curve of contact with K.) It is shewn that there exists a surface F of order 10 having D as double curve, and of which every adjoint sextic consists of Φ and a plane. *P. Du Val.*

Burau, Werner. Grundmannigfaltigkeiten der projektiven Geometrie. Collectanea Math. 3, 53–163 (1950).

This expository work is to consist of six parts, of which the present publication contains the first two. The author's intention is to give a general view of the most important algebraic-projective manifolds under large general classes. After a short introduction, outlining the basic ideas of projective geometry over the complex number field, Part I deals with Veronese Manifolds V_k^n, defined parametrically by equating the homogeneous coordinates in a space of suitably high dimensions to a maximal set of linearly independent n-ic forms in $k+1$ variables, in particular to the distinct monomials of this order; i.e., V_k^n is the projective model of the complete system of n-ic hypersurfaces in k dimensions. Thus V_1^n is the normal rational n-ic curve, V_2^2 is the Veronese surface, V_2^3 the nonic del Pezzo surface, and V_3^2 the octavic threefold with elliptic curve sections. These four are studied in considerable detail, and throughout it is the author's method to devote more attention to the particular cases with low values of the indices involved than to the general case.

Part II deals with Segre Manifolds $S_{i_1,\dots,i_s}$, again defined parametrically by equating coordinates to a maximal set of linearly independent forms, this time separately linear in s sets of $i_1+1, \cdots, i_s+1$ variables respectively; thus $S_{1,1}$ is the general quadric surface, $S_{n,1}$ is the $(n+1)$-ic $(n+1)$-fold generated by a rational ∞^1 of flat spaces, $S_{2,2}$ is Segre's sextic fourfold with elliptic curve sections. These are studied in detail, and attention is almost entirely confined to the case $s=2$, and particularly $S_{n,1}$ and $S_{n,n}$; just at the end some study is given to $S_{1,1,1}$ and $S_{1,1,\dots,1}$. In connexion with $S_{n,1}$ the general properties of the "Normregelgebilde" or normal manifolds generated by a rational ∞^1 of flat spaces are introduced; and these are also obtained as loci of joins of sets in an involution on a normal rational curve.

The method is a good and readable mixture of pure geometry with easy algebra; and the whole work seems to be both a valuable introduction to the subject, and also a useful reference for those who are not quite beginners.

P. Du Val (Bristol).

Du Val, Patrick. On surfaces whose canonical system is hyperelliptic. Canadian J. Math. 4, 204–221 (1952).

The author classifies the regular surfaces of genus $p\geqq 2$ whose canonical systems consist of irreducible hyperelliptic curves by describing their canonical $(p>2)$ and bicanonical models. For $p\geqq 7$ there is essentially a single general type for each value of p, but for $p<7$ there are numerous special cases. These are described in some detail, and the relations between them exhibited explicitly. The canonical model (for $p>2$) is a double rational surface R of order $p-2$ in $[p-1]$ (i. e., a rational normal scroll or a Veronese surface) with a branch curve of order $2n+4$ ($n+1$ being the linear genus) composed, in part, of $n-2p+4$ lines. The bicanonical model Φ is a double rational surface of order $2n$ in $[n+p]$ with $n-2p+4$ nodes which are isolated branch points, and in addition there is a branch curve of order $2n+4p$.

J. A. Todd (Cambridge, England).

Gaeta, Federico. Sull'esistenza di una serie infinita descontinua di trasformazioni razionali in sè sopra ogni superficie di genere lineare $p^{(1)}=1$ con un fascio di curve ellittiche di genere uguale alla irregolarità. Ist. Veneto Sci. Lett. Arti. Cl. Sci. Mat. Nat. 109, 135–139 (1951).

On a surface of the type specified in the title, let C be a general curve of the pencil, and G_d a set of d points on C, its intersection with a d-secant curve (d being the "determinant" of the surface.) The relation (equivalence on C)

$$nG_d \equiv X+(nd-1)P \quad (n=1, 2, \cdots)$$

determines a unique point X when P is given, and a set of $(nd-1)^2$ points P when X is given, and thus on the whole surface a rational self-transformation with each curve C self-corresponding. The sets of points P form an involution of order $(nd-1)^2$, with coincidences only at the Jacobian points of the pencil $\{C\}$, not generated by birational self-transformations of the surface. *P. Du Val* (Bristol).

Gaeta, Federico. Sur la limite inférieure l_0 des valeurs de l pour la validité de la postulation régulière d'une variété algébrique. C. R. Acad. Sci. Paris **234**, 1121–1123 (1952).

The limit in question is max $n_{ij}-n$, where n is the dimension of the space of the variety and n_{ij} are the degrees of the various forms of Ostrowski [Abh. Math. Sem. Univ. Hamburg 1, 281–326 (1922)] associated with the corresponding ideal. *J. A. Todd* (Cambridge, England).

Gaeta, Federico. Détermination de la chaîne syzygétique des idéaux matriciels parfaits et son application à la postulation de leurs variétés algébriques associées. C. R. Acad. Sci. Paris 234, 1833–1835 (1952).

Explicit forms are given for the chain of syzygies associated with the homogeneous ideal defined by the vanishing of the minors of maximum order in a homogeneous matrix.

J. A. Todd (Cambridge, England).

Room, T. G. Transformations depending on sets of associated points. Proc. Cambridge Philos. Soc. 48, 383–391 (1952).

The first part of this paper is devoted to plane involutory Cremona transformations which leave invariant each of a pencil of cubics u, with base points $A_1, \cdots, A_9$. If P, Q (lying on the same curve u) are corresponding points their residual point K depends only on u, and its locus as u varies is a rational curve unisecant to the pencil, outside its base points, i.e., of virtual grade -1 with respect to these. There is thus a one-one correspondence between these unisecant ("focal") curves ρ and the involutions R in question. The author points out that any row of nine integers (positive or negative) $m_1', \cdots, m_9'$ whose sum is $\equiv 2 \pmod 3$ determines uniquely such a curve, since, defining

$$6\mu = 9\sum m_i'^2 - (\sum m_i')^2 - 8, \quad 9m = \sum m_i' - 3\mu + 1,$$
$$m_i = m_i' - m,$$

the numbers $(\mu, m_1, \cdots, m_9)$ are positive integers satisfying the well-known conditions

$$3\mu = \sum m_i + 1, \quad \mu^2 = \sum m_i^2 - 1$$

for the unique μ-ic curve with m_i-ple point at A_i $(i = 1, \cdots, 9)$ to be a "focal" curve. Indicating such a curve by $\rho_{\mathbf{m}}$ and the corresponding transformation by $R_{\mathbf{m}}$ (the subscript $\mathbf{m}$ denotes the vector $(m_1, \cdots, m_9)$, but some of the results given later are only intelligible if it is allowed also to denote any vector such as $(m_1', \cdots, m_9')$ above, in which each element is increased by the same constant) and indicating a line by F and the neighbourhood of A_i also by A_i, so that $\rho_{\mathbf{m}} = \mu F - \sum_i m_i A_i$, it is stated without proof that

$$R_{\mathbf{m}} F = (3\mu + 5) F - \sum_i (2\mu + 1 - 3m_i) A_i$$
$$R_{\mathbf{m}} A_r = (2\mu + 1 - 3m_r) F - \sum_i (\mu - m_r - m_i) A_i - A_r$$

which means (as is easily verified by simple calculations) that if the transform of any curve $\mu_0 F - \sum m_{0i} A_i$ is $\bar{\mu} F - \sum \bar{m}_i A_i$ then

$$\begin{pmatrix} \bar{\mu} \\ \bar{\mathbf{m}} \end{pmatrix} = \mathbf{R}_{\mathbf{m}} \begin{pmatrix} \mu_0 \\ \mathbf{m}_0 \end{pmatrix} = (\mu \mathbf{F} - \sum m_i \mathbf{A}_i) \begin{pmatrix} \mu_0 \\ m_0 \end{pmatrix}$$

where

$$\mathbf{F} = \begin{pmatrix} 18 & -5\mathbf{v}' \\ 5\mathbf{v} & -\mathbf{\Omega} - 3\mathbf{I} \end{pmatrix}, \quad \mathbf{A}_i = \begin{pmatrix} 5 & -\mathbf{v}' - 3\boldsymbol{\varepsilon}_i' \\ \mathbf{v} + 3\boldsymbol{\varepsilon}_i & -\mathbf{I} - \mathbf{Y}_i \end{pmatrix}$$

in which $\mathbf{v}$ denotes a column of nine 1's, $\boldsymbol{\varepsilon}_i$ a column with 1 in the ith place and 0 elsewhere, $\mathbf{\Omega}$ a 9×9 matrix with all elements 1, $\mathbf{I}$ the unit 9×9 matrix, and $\mathbf{Y}_i$ a 9×9 matrix with 2 in the ith diagonal place, 1 in the other places of the ith row and column, and 0 elsewhere. These matrices satisfy

$$\mathbf{R}_{\mathbf{m}}^2 = \mathbf{I}, \quad \mathbf{R}_{\mathbf{m}} \mathbf{\Gamma} = \mathbf{\Gamma} \mathbf{R}_{\mathbf{m}}', \quad \mathbf{R}_{\mathbf{m}_1} \mathbf{R}_{\mathbf{m}_2} \mathbf{R}_{\mathbf{m}_3} = \mathbf{R}_{\mathbf{m}_1 - \mathbf{m}_2 + \mathbf{m}_3},$$
$$\mathbf{R}_{\mathbf{m}} \mathbf{U} = \mathbf{U}, \quad R_{\mathbf{m}} \rho_{\mathbf{m}'} = \rho_{-\mathbf{m}-\mathbf{m}'}$$

where

$$\mathbf{\Gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{I} \end{pmatrix}$$

is the intersection matrix of the base $(F, A_1, \cdots, A_9)$, and $\mathbf{U} = 3\mathbf{F} - \sum \mathbf{A}_i$, i.e., corresponds to the cubics u in the same way as $\mathbf{R}_{\mathbf{m}}$ to $\rho_{\mathbf{m}}$.

In the second part similar consideration is given to the (s, s) transformation (not $(1, s)$ as the author states) in which P, Q correspond if they are on the same curve u, and the third intersection of PQ with this cubic is on a fixed curve which has now not 1 but s variable intersections with the variable u. These transformations have base points not only in $A_1, \cdots, A_9$ but the 12 nodes $D_1, \cdots, D_{12}$ of cubics of the pencil, and the corresponding matrices are accordingly 22×22. These matrices are found in only two cases, in which the s-secant curve is respectively the locus of points of contact of tangents to u from A_1 (a quintic with triple point at A_1 passing simply through $A_2, \cdots, A_9$ and $D_1, \cdots, D_{12}$) and the locus of inflexions of u (a 12-ic passing triply through $A_1, \cdots, A_9$ and doubly through $D_1, \cdots, D_{12}$).

The third part deals with the involutions in space similarly determined by the twisted quartics through 8 associated points, i.e., such that if P, Q correspond they lie on the same quartic of the system, and PQ meets the tangent to this cubic at its intersection with a unisecant surface. These are punctual transformations, i.e., the homaloidal system corresponding to planes G, and the transforms of the eight points $B_1, \cdots, B_8$, are completely determined by their order ν and multiplicities $\nu_1, \cdots, \nu_8$ in these points. The theory is closely similar to that in the first part, the matrices $\mathbf{F}, \mathbf{A}_i, \mathbf{\Gamma}$, being replaced by

$$\mathbf{G} = \begin{pmatrix} 76 & -36\mathbf{v}' \\ 18\mathbf{v} & -8\mathbf{\Omega} - 4\mathbf{I} \end{pmatrix}, \quad \mathbf{B}_i = \begin{pmatrix} 15 & -6\mathbf{v}' - 8\boldsymbol{\varepsilon}_i' \\ 3\mathbf{v} + 4\boldsymbol{\varepsilon}_i & -\mathbf{\Omega} - \mathbf{I} - 2\mathbf{Y}_i \end{pmatrix},$$

$$\mathbf{\Delta} = \begin{pmatrix} 2 & 0 \\ 0 & -\mathbf{I} \end{pmatrix}$$

where $\mathbf{v}, \boldsymbol{\varepsilon}_i, \mathbf{\Omega}, \mathbf{I}, \mathbf{Y}_i$ have the same meaning as before except that they are columns of 8 elements or 8×8 matrices. (The author says that $\mathbf{\Delta}$, unlike $\mathbf{\Gamma}$, is not an intersection matrix; it is however the intersection matrix of the curves traced by the base $(G, B_1, \cdots, B_8)$ on a general quadric through the eight points.)

As almost all the theorems are stated with the minimum of proof, or with none at all, the paper is exceptionally hard to read.

P. Du Val (Bristol).

Petrovskiĭ, I. G., and Oleĭnik, O. A. On the topology of real algebraic surfaces. Amer. Math. Soc. Translation no. 70, 20 pp. (1952).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 389–402 (1949); these Rev. 11, 613.

Galafassi, Vittorio Emanuele. Il segno del risultante d. più forme reali nella topologia degli spazi proiettivi reali Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 275–289 (1950).

In projective space S_r, the resultant R of $r+1$ algebraic forms F_i of orders n_i respectively, is the condition that the $r+1$ forms have at least one point in common. The author obtains a topological characteristic defining the sign of R when at least two n_i are even. He shows that $R > 0$ or $R < 0$ according as this characteristic has the value 1 or 0, mod 2, respectively. The cases $r = 1$ and $r = 2$ are discussed fully and the method of generalizing the result for any r is illustrated by the case $r = 3$.

T. R. Hollcroft (Aurora, N. Y.).

Nidito, Maria Mehle. Sulla classificazione cremoniana delle congruenze di coniche di indice 1 dell' S_3. Rend. Sem. Mat. Univ. Padova 20, 430–445 (1951).

D. Montesano [Rend. Accad. Sci. Fis. Mat. Napoli (3) 1, 93–110, 155–181 (1895)] a résolu le problème de la classification des congruences de coniques d'indice 1 de l'espace projectif à trois dimensions, en se basant sur le type de l'involution déterminée par les coniques de la congruence sur un plan générique de l'espace, les points conjugués dans cette involution étant ceux situés sur une même conique. S. Kantor [Amer. J. Math. 23, 1–28 (1901)] affirme, sans démonstration, que parmi les différents types projectivement distincts (au nombre de 25) de la classification de Montesano, deux seulement sont crémoniennement distincts, et, dans une brève critique de la méthode de classification adoptée par Montesano, il reproche à celle-ci de ne pas avoir le caractère crémonien. L'auteur revient sur le travail de Kantor et lui apporte des précisions nécessaires. Il montre quels sont, parmi les types de congruences de coniques projectivement distincts de la classification de Montesano, ceux qui sont réductibles à des étoiles (de droites) ou crémoniennement équivalents à des types plus simples. Il établit que parmi les 25 types de congruences de Montesano deux seulement sont crémoniennement distincts, à savoir: La congruence constituée par ∞^1 faisceaux de coniques dont les plans forment un faisceau linéaire, et celle des coniques caractéristiques d'un réseau de surfaces cubiques admettant une courbe base du 7ème ordre. La démonstration du fait que les deux types précédents sont crémoniennement distincts est basée sur la considération des deux systèmes de coniques dégénérées des deux congruences: ces deux systèmes sont crémoniennement distincts et il en est par suite de même des deux congruences dont ils font respectivement partie. *P. Vincensini* (Marseille).

Abellanas, Pedro. Algebraic correspondences. II. Revista Mat. Hisp.-Amer. (4) 11, 159–179 (1951).

Let T be an irreducible algebraic correspondence between varieties V and V'. If W is a subvariety of V which is regular for T, then the transform $T[W]$ is unmixed, and formulas are given relating the dimensions of W and $T[W]$. (Regularity of W for T is meant in the sense previously defined by the author [same Revista (4) 9, 175–233 (1949); these Rev. 12, 740].) A general hyperplane section of V is regular for T, and there exist hyperplanes with coefficients in the ground field which cut out on V a regular section. Finally certain local properties of T are studied. The proofs are ideal-theoretic. *I. S. Cohen* (Cambridge, Mass.).

Abellanas, Pedro. Algebraic correspondences. II. Revista Mat. Hisp.-Amer. (4) 11, 138–158 (1951). (Spanish)

A Spanish version of the paper reviewed above.

Severi, Francesco. Fondamenti per la geometria sulle varietà algebriche. II. Ann. Mat. Pura Appl. (4) 32, 1–81 (1951).

The previous paper with the same title is the famous paper of 1909 [Rend. Circ. Mat. Palermo 28, 33–87 (1909)], which was the first serious attempt to extend the ideas of the birational geometry of surfaces to varieties of higher dimension. The present work is in part a critical review of the progress of this theory since 1909, and in part a reformulation of the theory, shorn of the unproved assumptions of the earlier work. The central result of the paper is a proof of the relative invariance of the arithmetic genus, according to either of the two classical definitions.

The author is concerned throughout with a non-singular variety V_d in S_r, or its general projection V_d' into S_{d+1}. The first part of the paper deals with the definition of the (impure) canonical system, which is obtained in the general form $W-\sum_{h=1}^{i}(d_h+1)A^h$, where $|A^h|$ is a linear system of dimension d_h (with $\sum d_h=d$), which is free to vary in a linear system of dimension $\geqq d$ which is free from base points and fundamental varieties, and W is the Jacobian variety of the systems $|A^h|$. The relative invariance of this system, and its independence of the integers d_h is demonstrated. Next, the idea of the pure canonical system is introduced on a variety of positive geometric genus, and its absolute invariance established.

The author next turns to the arithmetic genus. As is well-known, two characters can be defined for a V_d; the postulation P_d^* of the double variety of V_d' for the adjoint primals which cut out the canonical system, and an expression p_d which is a linear combination of the coefficients which occur in the postulation formula for V_d itself. For $d=1$, 2, these numbers are known to be equal to each other, and to be absolute invariants. The author reviews earlier incomplete attempts to obtain corresponding results for $d>2$, referring in particular to the author's paper of 1909 already mentioned, Albanese [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 33, 179–182, 210–214 (1924)] and the reviewer [Proc. London Math. Soc. (2) 43, 190–225 (1937)]. All this work was based on certain postulates. The author accordingly proceeds to consider the whole question de novo. He starts by considering the character p_d, which he terms the "effective arithmetic genus". If $|A|$ is a linear system on V_d, of freedom $\geqq d$, free from base points and with irreducible characteristic systems, the effective characters of $|A|$ are its degree n_0 and the effective i-dimensional arithmetic genus n_i, which is the effective arithmetic genus of $|A^{d-i}|$ $(i=1, \cdots, d-1)$. If $|A|$, $|B|$, $|C|$ are three such linear systems such that $|C|=|A+B|$ we have

$$g(C)=g(A)+g(B)+g(AB),$$

where $g(A)$ is the effective arithmetic genus of A. The author proceeds to compute $g(C^i)$ for $i>1$, and to extend the formula to the sum of several linear systems and to multiples of a linear system, and finally to obtain a definition for virtual characters of a virtual V_{d-1} on V_d. If, now, A is a V_{d-1} (possibly virtual, but pure) on V_d, the author defines

$$\delta(A)=\sum_{\alpha=1}^{d}(-1)^{d-\alpha}g(A^\alpha)+(-1)^d p_d+d$$

to be the "virtual dimension" of $|A|$. He proves that if A, B, C are virtual V_{d-1} such that $C=A+B$, then $\delta(C)=\delta(A)+\delta(\overset{*}{B}, C)+1$, where the asterisk indicates that (B, C) is to be considered on the virtual variety B. A complete linear system of V_{d-1} on V_d is "regular" if its effective dimension is equal to its virtual dimension. The author proves that the section of V_d by primals of a sufficiently high order through a given irreducible non-singular V_{d-1} on V_d is regular. The relative invariance of p_d is now deduced, by induction on d, as follows. Let V_d and $\bar{V}_d$ be two varieties in birational correspondence without exceptional elements, let $|E|$, $|\bar{F}|$ be the linear systems on $\bar{V}_d$ and V_d. It is shown that for sufficiently large l the complete linear systems $|G|=|l(E+F)|$ and $|\bar{G}|=|l(\bar{E}+\bar{F})|$ are regular. Since these systems, and their characteristic systems, correspond in the birational correspondence between V_d and $\bar{V}_d$ it follows, on writing down the expressions for $\delta(G)$ and $\delta(\bar{G})$ and using the inductive hypothesis, that $p_d(V_d)=p_d(\bar{V}_d)$.

The relative invariance of P_d^a is then deduced by establishing the relation

$$\omega_0-\omega_1+\omega_2+\cdots+(-1)^{d-1}\omega_{d-1}+d+(-1)^{d-1} = p_d^a+(-1)^{d-1}p_d,$$

where the ω's are the virtual characters of the canonical system, and are themselves relative invariants. The author finally indicates, without details, the lines along which he hopes to establish the relation $p_d=P_d^a$. *J. A. Todd.*

Segre, Beniamino. L'anneau d'équivalence sur une variété algébrique. C. R. Acad. Sci. Paris **234**, 1663–1665 (1952).

Segre, Beniamino. Variétés covariantes d'immersion et variétés canoniques sur une variété algébrique ou topologique. C. R. Acad. Sci. Paris **234**, 1731–1733 (1952).

Let V be an irreducible algebraic variety of v dimensions. The equivalence classes of subvarieties of V form a ring $\mathfrak{A}_V$, with V as unit element and intersection as the operation of multiplication. Let P be a subvariety of V, pure and of dimension p. If $P_0=P, P_1, P_2, \cdots$ are pure virtual varieties, such that P_i belongs to $\mathfrak{A}_P$ and has dimension $p-i$, the sequence $\{P\}=\{P_0, P_1, \cdots\}$, which is called a sequence of support P (suite de support P) defines an element of $\mathfrak{A}_P$, the direct sum of the elements of $\mathfrak{A}_P$ corresponding to $P_0, P_1, \cdots$, and this element has an inverse in $\mathfrak{A}_P$ which is another sequence $\{\bar{P}\}$ of the same type. If M, N, T are pure varieties on V, such that $T\subset M$, $T\subset N$, of dimensions m, n, t, where $t\geqq m+n-v$, these define an element $S=(MN)_V^T$ of dimension $m+n-v$ such that, if M and N are represented by varieties intersecting simply along T, S is their residual intersection. The symbol $(M^{(1)}M^{(2)}\cdots M^{(s)})_V^T$ can be defined in a similar manner.

These definitions occupy the first of the two papers under review. In the second paper, the author defines certain covariants of immersion of one variety in another. Let P be a pure subvariety of V of dimension p, let $A^{(1)}, \cdots, A^{(s)}$ be arbitrary hypersurfaces on V, and let $V_i(A^{(1)}, \cdots, A^{(s)})$ be the sum of their virtual intersections taken i at a time. A sequence of support P, $\{P_V\}=\{P_{V,0}, P_{V,1}, \cdots\}$ is defined, for each s such that $v-p\leqq s\leqq v$, by the property that

$$(A^{(1)}\cdots A^{(s)})_V-(A^{(1)}\cdots A^{(s)})_V^P=\sum_{i=0}^{t}P_{V,i}V_{t-i}(A^{(1)}\cdots A^{(s)})$$

holds identically in $A^{(1)}\cdots A^{(s)}$; here $t=s+p-v$. Among other applications of these ideas the following is significant. Let V be non-singular and let $W=V\times V$. The diagonal cycle on W is birationally equivalent to V and may be denoted by the same symbol. Let $\{V_W\}$ be the sequence of support defined as above, and $\{\bar{V}_W\}$ the sequence corresponding to the inverse element of $\mathfrak{A}_V$. Then the ith canonical variety on V is given by $(-1)^i\bar{V}_{W,i}$. *J. A. Todd.*

Samuel, Pierre. La notion de multiplicité en algèbre et en géométrie algébrique. J. Math. Pures Appl. (9) 30, 159–205 (1951).

Samuel, Pierre. La notion de multiplicité en algèbre et en géométrie algébrique. II. J. Math. Pures Appl. (9) 30, 207–274 (1951).

Many of the results of these two papers have been announced without proof in four notes [C. R. Acad. Sci. Paris 225, 1111–1113, 1244–1245 (1947); 228, 158–159, 292–294 (1949); these Rev. 9, 325; 10, 732]. The present review should be read in conjunction with these four reviews, whose contents will not be repeated here and whose notation and terminology will be used without further explanation.

Chapter I of the present work proves the theorems announced in the first note just referred to. The results of the second note are proved in Chapter II. In addition ($\mathfrak{O}/\mathfrak{m}$ always being assumed infinite): If $\mathfrak{q}$ is primary for $\mathfrak{m}$, there exists a primary ideal $\mathfrak{q}'\subset\mathfrak{q}$ generated by a system of parameters and such that $e(\mathfrak{q}')=e(\mathfrak{q})$; this was announced in the second note, but with a restrictive hypothesis on $\mathfrak{O}$. For the geometrical applications it is important that when $\mathfrak{O}$ is a quotient ring on an algebraic variety these parameters can be selected as appropriate linear forms in an arbitrary set of generators of $\mathfrak{q}$. If $\mathfrak{q}$ is of length m and is generated by a system of parameters, then $m\geqq e(\mathfrak{q})$; under certain conditions equality holds and $P_{\mathfrak{q}}(n)=m\binom{n+d}{d}$ for all n ($d=\dim\mathfrak{O}$). Chevalley's notion of a geometric local ring [Trans. Amer. Math. Soc. 57, 1–85 (1945); these Rev. 7, 26] is generalized, and the following is proved about such a ring $\mathfrak{O}$: Let $\mathfrak{q}$ be primary in $\mathfrak{O}$ for a prime ideal $\mathfrak{p}$, let $\mathfrak{O}^*$ be the completion of $\mathfrak{O}$, $\mathfrak{p}^*$ a prime ideal of $\mathfrak{O}^*\mathfrak{p}$, $\mathfrak{q}'$ the primary component of $\mathfrak{O}^*\mathfrak{q}$ belonging to $\mathfrak{p}^*$. Then $\mathfrak{q}'$ and $\mathfrak{q}$ have the same length. A special case of this gives Chevalley's "theorem of transition" [loc. cit., p. 22]. In the Kronecker product of local rings, the multiplicity is shown to be multiplicative in a certain simple sense.

The author's definition of multiplicity is more general than Chevalley's in that it is not restricted to primary ideals generated by systems of parameters, nor to equidimensional local rings. It coincides with the latter when the latter applies, as was pointed out in the review of the second paper cited above. Now that the proofs are available, it appears that they are in general simpler than the ones using the earlier definition.

Chapter III contains some preliminary geometric applications. The multiplicity of a hypersurface H at a point P is shown to be equal to $e(\mathfrak{m})$, $\mathfrak{m}$ being the maximal ideal in the quotient ring of P on H. Using his own and Chevalley's methods, the author proves several known theorems. The chapter closes with a discussion which clarifies the relation between the intersection theories of Chevalley [loc. cit.] and Weil [Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., vol. 29, New York, 1946; these Rev. 9, 303].

Chapter IV is mostly devoted to a solution of the problem posed by Weil [loc. cit.] of constructing a theory of specialization of cycles of arbitrary dimension. Specialization of divisors in projective n-space (i.e., in effect, of hypersurfaces) is defined simply by specialization of the coefficients of the equation of the hypersurface. If, now, X and X' are cycles of dimension $r<n-1$, then X' is called a specialization of X if the projection of X' on a generic $(r+1)$-dimensional subspace is a specialization of the projection of X. It is then proved that Weil's requirements are satisfied. Other topics in this chapter are: (1) a proof of Severi's theorem that every r-dimensional cycle in n-space is an intersection of $n-r$ divisors; (2) proofs that certain "undesirable" phenomena are of algebraic character; for example, if V is a d-dimensional variety in n-space, the set of linear subspaces L^{n-d-1}, the projection of V from which is of degree >1, is an algebraic system.

Chapter V concerns the intersection multiplicity for improper components of intersection and gives proofs of the results announced in the third note cited above. (The review of this note contains a misprint: In line 1 of paragraph 2, read M^{Δ} for M.) If U is a subvariety of a variety V the

multiplicity $m(U; V)$ of U on V is defined as $e(\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal in the quotient ring of U on V (cf. the theorem on hypersurfaces referred to in Chapter III). It is proved that $m(U; V)=i(U; U\cdot V)=i(U; \bar{U}\cdot V)$, where $\bar{U}$ is a generic cylinder passing through U. If P is a point of V, a line L through P is said to be tangent if $i(P; L\cdot V)>m(P; V)$; the set of all tangent lines is an algebraic manifold. The associativity formula for intersections of three varieties is proved only under certain restrictive hypotheses; several counterexamples show that it fails in general.

Chapter VI gives the proofs of the results stated in the fourth note mentioned above. (In line 15 of the review of this note, there is a misprint: read $w-s-t$ for $w-r-s$.)

The introduction to the present work gives a list of twelve unsolved problems. *I. S. Cohen* (Cambridge, Mass.).

Behrens, Ernst-August. Zur Schnittmultiplizität uneigentlicher Komponenten in der algebraischen Geometrie. Math. Z. 55, 199–215 (1952).

When the dimension p of a component P of the intersection of two algebraic varieties A^a and B^b in projective space S_n is such that $p=a+b-n$, P is said to be proper; when $p>a+b-n$, P is said to be improper. The author extends to improper components the notion of intersection multiplicity $\mu(A\cdot B, P)$ in the case P is a point. By definition $\mu(A\cdot P, P)$ is the classical intersection multiplicity $i((A\times B)\cdot M, P\times P; S_n\times S_n)$ [cf. A. Weil, Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; these Rev. 9, 303], where M denotes the variety of pairs (x, y) of points of S_n such that the line xy meets a generic linear $(n-a-b-1)$-dimensional variety L of S_n. If we denote by A' a generic cone of dimension $a+b-n$ having A as a base, then P is a proper component of $A'\cap B$ and $\mu(A\cdot B, P)=i(A'\cdot B, P; S_n)$, and the given definition coincides with one suggested by F. Severi [Abh. Math. Sem. Univ. Hamburg 9, 335–364 (1933)]. Further properties of the μ-symbol are given: commutativity, criterion for multiplicity 1, biregular invariance. This paper deals only with components of dimension zero; however, counter-examples to the associativity property of intersections are given. A similar theory and more complete results were previously given by the reviewer [C. R. Acad. Sci. Paris 222, 158–159 (1949); these Rev. 10, 732; and the paper reviewed above]. *P. Samuel* (Ithaca, N. Y.).

Chow, Wei-Liang. On the fundamental group of an algebraic variety. Amer. J. Math. 74, 726–736 (1952).

Let ϕ be a rational mapping of an algebraic variety U on a variety V. The system of varieties $\phi^{-1}(y)$ $(y\epsilon V)$ do not form a fibre bundle over U, but has some properties similar to properties of a fibre bundle. It forms a fibre system. Some properties of fibre systems are established, including the covering homotopy theorem. The results obtained are used to establish two theorems: (a) the Severi-Lefschetz theorem that any 1-cycle on a variety U is homotopic to a cycle on any subvariety G of U which belongs to an algebraic system which covers U, provided the system has at least one base point; (b) if ϕ is a rational mapping $U\rightarrow V$ such that $\phi^{-1}(\eta)$ is irreducible, where η is a generic point of V, and if the points y of V for which the specialisation $\phi^{-1}(\eta)\rightarrow\phi^{-1}(y)$ is not uniquely determined, or has multiple components, is of dimension $\leqq\dim V-2$, then the fundamental group of U can be mapped homomorphically on the fundamental group of V, the kernel of the homomorphism being the image induced by the identical mapping into U of the fundamental group of $\phi^{-1}(y_0)$, where y_0 is a sufficiently general point of V. *W. V. D. Hodge* (Cambridge, England).

Kodaira, Kunihiko. The theorem of Riemann-Roch on compact analytic surfaces. Amer. J. Math. 73, 813–875 (1951).

In this paper a Riemann-Roch theorem is established for an arbitrary curve Γ (positive divisor without multiple components) on a compact Kähler surface. The method is based on de Rham's theory of currents, and turns on two fundamental existence theorems, the first of which is as follows. Let T be a $(p+1)$-current of type $(p, 1)$ defined over the whole of a compact Kähler manifold $M=M^n$ of arbitrary (complex) dimension n, and suppose that T is closed and has a vanishing harmonic component. Then $\Theta=(d\Lambda+i\delta)GT$, where Λ is the Hodge operator and G the Green's operator of de Rham, is a current of type $(p, 0)$ which satisfies $d\Theta=iT$. In particular, Θ is a holomorphic differential of degree p in $M-|T|$, $|T|$ the carrier of T. A proof of this theorem (the existence of G assumed) is immediate.

The second existence theorem, due to A. Weil, asserts the existence of an analytic cross-section in a certain bundle F over a compact curve. Let M be a complex manifold with a locally finite open covering by coordinate neighborhoods $\{U_j\}$ such that each intersection $U_j\cap U_k$ is a cell (if non-empty). Let there be given a system $\{t_{jk}\}$ of functions t_{jk} where t_{jk} is holomorphic and non-vanishing in $U_j\cap U_k$ and $t_{jk}t_{kj}=1$ in $U_j\cap U_k$, $t_{jk}t_{kn}t_{nj}=1$ in $U_j\cap U_k\cap U_n$. If we take the complex projective line S_1 as fibre, M as base space, the t_{jk} as coordinate transformations, we obtain a complex coordinate bundle with bundle space F and with the multiplicative group of complex numbers as group. The existence of an analytic cross-section of F is equivalent to the existence of a system $\{h_j\}$ of functions h_j, h_j meromorphic in U_j, $h_j/h_k=t_{jk}$. The theorem of Weil asserts the existence of an analytic cross-section in the special case in which M is a compact curve.

Let E be the corresponding principal bundle (whose fibre is the multiplicative group of complex numbers), and let u (cohomology class of M of dimension 2) be the topological obstruction for the triviality of E. The reviewer remarks that, if M is compact Kähler, an analytic cross-section of F exists in the general n-dimensional case if and only if the $(2n-2)$-dimensional homology class dual to u is represented by an analytic cycle (a condition trivially satisfied if $n=1$).

Now let Γ be a curve on the Kähler surface M. Then Γ is decomposed into a sum $\Gamma=\sum\Gamma_\nu$ of finitely many irreducible curves Γ_ν each of which is a holomorphic image of a non-singular model $\mathbf{\Gamma}_\nu$: $\Gamma_\nu=\varphi(\mathbf{\Gamma}_\nu)$. The direct sum $\mathbf{\Gamma}=\sum\mathbf{\Gamma}_\nu$ constitutes the non-singular model of Γ. Points of $\mathbf{\Gamma}$ are denoted by $p, q, \cdots$, and points of M by $\mathfrak{p}, \mathfrak{q}, \cdots$. The mapping $q\rightarrow\mathfrak{q}=\varphi(q)$ from $\mathbf{\Gamma}$ onto Γ is one-one unless $\mathfrak{q}$ is a singular point in which case the inverse image consists of a finite number of points each of which corresponds to an irreducible branch of the curve Γ passing through $\mathfrak{q}$. Let τ_p be a local uniformization variable on $\mathbf{\Gamma}$ with the center p, and let (z^1, z^2) be a system of local coordinates on M with origin $\mathfrak{p}=\varphi(p)$. Then the mapping $q\rightarrow\mathfrak{q}=\varphi(q)$ which maps the neighborhood U_p of p onto the branch Γ_p can be written $z^1=\varphi^1(\tau_p)$, $z^2=\varphi^2(\tau_p)$. At each point $\mathfrak{p}\epsilon M$ the curve Γ is represented by a minimal local equation $R_{\mathfrak{p}}(z^1, z^2)=0$. Consider two points p, q on $\mathbf{\Gamma}$ such that $U_p\cap U_q\neq 0$. The ratio $T_{pq}=R_p(z)/R_q(z)$ is a non-vanishing holomorphic function of $z=(z^1, z^2)$ in some neighborhood of $\varphi(U_p\cap U_q)$, and $t_{pq}(\tau)=T_{pq}(\varphi^1(\tau), \varphi^2(\tau))$ defines a system $\{t_{pq}\}$ satisfying the

above requirements. Weil's theorem establishes the existence of a system $\{h_p\}$, $h_p/h_q=t_{pq}$, and if $V_p(h_p)$ is the order of h_p at p, $\mathfrak{b}=\sum V_p(h_p)\cdot p$ is a divisor on Γ. The divisor class $\{\mathfrak{b}\}$ of $\mathfrak{b}$ is uniquely determined by Γ and is called its characteristic divisor class. It is shown that the degree of $\mathfrak{b}$ is equal to the topological intersection number of the cycle Γ with itself. Let $\sigma_p=\partial\varphi_p^2/\partial_1R_p(\varphi_p)=-\partial\varphi_p^1/\partial_2R_p(\varphi_p)$ where $\partial_\alpha=\partial/\partial s^\alpha$ $(\alpha=1, 2)$, and set $\mathfrak{c}=-\sum V_p(\sigma_p)\cdot p$. The "conductor" $\mathfrak{c}$ is a positive divisor on Γ carried on the inverse image of the set consisting of all singular points of Γ.

Let $\mathfrak{A}(\Gamma)$ denote the space of additive meromorphic functions on M which are multiples of $-\Gamma$, and let $\mathfrak{F}(\Gamma)$ be the subspace of single-valued functions. The system $\{h_p\}$ makes it possible to define for each $F\epsilon\mathfrak{A}(\Gamma)$ a Severi residue $f=R_pF(\varphi_p)/h_p$ on Γ satisfying the following two conditions: (α) $(f)+\mathfrak{b}\geqq 0$; (β) $\sum_{\varphi(p)=\mathfrak{p}} \operatorname{Res}_p [(\varphi_p^1)^k(\varphi_p^2)^l fh_p\sigma_p]=0$ $(k, l=0, 1, 2, \cdots)$ where Res_p denotes the residue on Γ at p. Let $\mathfrak{f}(\Gamma, \mathfrak{b})$ be the space of all meromorphic functions f on Γ satisfying (α) and (β), and let

$$Q_f[\Psi]=-\int_\Gamma \sum_\beta \Psi_{12\bar\beta}(\varphi_p)fh_p\sigma_p d\varphi_p^{\bar\beta},$$

$f\epsilon\mathfrak{f}(\Gamma, \mathfrak{b})$. Then Q_f is a 1-current of type $(0, 1)$ and dQ_f is a 2-current of type $(1, 1)$. By the first existence theorem, $\Theta_f=(d\Lambda+i\delta)GdQ_f$ is a holomorphic differential of degree 1 in $M-\Gamma$, and it is shown that $F=2\pi\int^z\Theta_f$ belongs to $\mathfrak{A}(\Gamma)$. A correspondence is thus set up between the spaces $\mathfrak{A}(\Gamma)$ and $\mathfrak{f}(\Gamma, \mathfrak{b})$ and, if $\omega=ig_{\alpha\bar\beta}dz^\alpha dz^{\bar\beta}$ is the fundamental form, $f\epsilon\mathfrak{f}(\Gamma, \mathfrak{b})$ corresponds to a function $F\epsilon\mathfrak{F}(\Gamma)$ if and only if $Q_f[\omega\cdot A]=0$ for all simple differentials A of the first kind on M. The problem of determining the dimension of $\mathfrak{F}(\Gamma)$ is therefore reduced to that of determining the dimension of $\mathfrak{f}(\Gamma, \mathfrak{b})$, and in this manner a Riemann-Roch formula is obtained.

In a later section of the paper the same idea is applied to obtain the dimension of the space $\mathfrak{W}(\Gamma)$ composed of double differentials on M which are multiples of $-\Gamma$, but in this case the Severi residue of a function is replaced by the Poincaré residue $\xi=\mathfrak{R}(W)=R_pW_{12}(\varphi_p)\sigma_p$ of a double differential. If $W\epsilon\mathfrak{W}(\Gamma)$, then (in the sense of currents) $dW=2\pi i\mathfrak{R}(W)$.

The main idea of the author is to extend a current carried on the curve Γ (residue) into a current carried on the whole manifold M by means of the Green's operator G, the latter current being holomorphic in $M-\Gamma$. The development of this idea is accomplished by the author with masterful technique. *D. C. Spencer* (Princeton, N. J.).

Kawahara, Yûsaku. On the differential forms on algebraic varieties. Nagoya Math. J. 4, 73–78 (1952).

It is proved that a differential form ω of the first kind on a complete variety U of dimension n (perfect field of definition) induces on a simple subvariety V a differential form of the first kind (Theorem 2). The proof requires a characterization of ω by means of the prime divisors of the field $k(P)$, P a generic point of U, and the realization of the center of an $(n-2)$-dimensional rank-2 valuation of a regular field $k(P)$ on a model of $k(P)$ by a simple $(n-2)$-dimensional subvariety, necessitating Zariski's theory of local uniformization [e.g., Ann. of Math. 45, 472–542 (1944); these Rev. 6, 102]. Furthermore, the author shows that the assumption on V may be dropped provided k has the characteristic 0 (Theorems 3 and 4). A counterexample is quoted which shows that the latter condition may not be dropped in general. *O. F. G. Schilling* (Chicago, Ill.).

Differential Geometry

Horninger, H. Über eine Evolventenschraubung (zylindrische Schrotung einer Ebene). Rev. Fac. Sci. Univ. Istanbul (A) 26, 255–277 (1951). (German. Turkish summary.)

If a plane rolls on a fixed (circular) cylinder the resulting motion is called a plane involute motion (planare Evolventenbewegung). If, in addition, the system is subjected to a uniform translation in the direction of the axis, the motion is called an involute screw motion. The author here studies the curves and surfaces generated by this motion. It is an extension of an earlier paper by the author [Bull. Tech. Univ. Istanbul 3, no. 1, 103–122 (1951); these Rev. 13, 580]. *S. B. Jackson* (College Park, Md.).

Strubecker, Karl. Elliptische Schraubungen und nichteuklidische Loxodromen. Mat. Tidsskr. B. 1951, 71–76 (1951).

This is based on a comprehensive account of cycloidal curves by Fabricius-Bjerre [Danske Vid. Selsk. Mat.-Fys. Medd. 26, no. 9 (1951); these Rev. 13, 275]. The theorem he proves concerns the polars of cycloidal curves called "Ährenkurven", which Fabricius-Bjerre had proved could be represented as non-Euclidean loxodromic curves, the proof consisting of a dualizing of a theorem due to Wunderlich that any circular cycloid can be represented as a non-Euclidean tractrix. In this paper the author proves that the loxodromic character of the "Ährenkurven" can be obtained directly from the results of two papers of his on non-Euclidean screw motions [Monatsh. Math. Phys. 38, 63–84 (1931); Akad. Wiss. Wien, S.-B. IIa. 139, 421–450 (1930)]. *E. T. Davies* (Southampton).

Fabricius-Bjerre, Fr. Note on a theorem of G. Bol. Arch. Math. 3, 31–33 (1952).

The author generalizes the six-vertex-theorem of affine [or projective] differential geometry and the classical four-vertex-theorem to convex arcs whose end-points have the same osculating conic, respectively, circle. As he points out himself, his results are essentially anticipated by Mukhopadhyaya [Math. Z. 33, 648–662 (1931)]. *P. Scherk.*

Hjelmslev, Johannes. Ein Satz über monotone Raumkurven im R_n mit einer Anwendung auf elliptisch und hyperbolisch gekrümmte Ovale. Acta Math. 87, 59–82 (1952).

Consider any simple bounded arc (curve) in projective n-space R_n. A set of $n+1$ points on such an arc (curve) may be arranged in a natural sequence order along the arc (curve). If any two such sequences of $n+1$ points, when similarly directed, give the same orientation in R_n, the arc (curve) is said to be monotone. A curve is called pointwise monotone if every point is an interior point of a monotone subarc. If attention is confined to curves having no subarc in any linear subspace, the condition of being monotone is equivalent to assuming that the curve is met in at most n points by every hyperplane R_{n-1}. An arc is called ordinary if it has sufficient regularity to have uniquely defined osculating linear spaces and half-spaces of dimensions up to $n-1$. The principal result of the first half of the paper is the following theorem: Every closed, pointwise monotonic, ordinary curve in projective n-space R_n, which contains no n linearly dependent points, is monotone in the large.

This fundamental theorem is then applied to establish results of Böhmer, Mohrmann, and Haupt on elliptically

and hyperbolically convex plane curves, and parabolic convexity and concavity. This is achieved by mapping the projective plane into the so-called Veronese surface in R_5. By this mapping the conics are carried into the intersections of the Veronese surface with the hyperplanes R_4.

As noted in the introduction the material of this paper is taken from manuscripts and notes left by Hjelmslev, which have been arranged and edited by Fabricius-Bjerre.

S. B. Jackson (College Park, Md.).

Sz.-Nagy, Gyula. Tschirnhaus'sche Flächen und Kurven. Acta Math. Acad. Sci. Hungar. 1, 167–181 (1950). (German. Russian summary)

Given n points $F_1, \cdots, F_n$ in the plane (respectively, space). Each point F_k is provided with a real weight q_k. Let $r_k = r_k(P)$ denote the distance of the point P from F_k. The locus of those points P for which $C = \sum_1^n q_k r_k$ is constant is a Tschirnhaus curve (respectively, surface) $T_n = T_n(C)$. In a previous paper [same Acta 1, 36–45 (1950); these Rev. 12, 733], the author discussed the T_n's with positive weights. They are convex. In the present paper some simple properties of the general T_n's are collected. The last section deals with the case $n=2$. *P. Scherk* (Los Angeles, Calif.).

Scherrer, W. Zur elementaren Flächentheorie. Comment. Math. Helv. 26, 78–80 (1952).

Feld, J. M. On the geometry of lineal elements on a sphere, Euclidean kinematics, and elliptic geometry. Canadian J. Math. 4, 93–110 (1952).

The geometry of turns and slides was initially studied by Kasner [Amer. J. Math. 33, 193–202 (1911)], followed by a series of papers by Kasner and De Cicco [see De Cicco, Trans. Amer. Math. Soc. 47, 207–229 (1940); Kasner and De Cicco, Proc. Nat. Acad. Sci. U. S. A. 37, 224–225 (1951); these Rev. 1, 170; 13, 71]. Recently A. Narasinga Rao, K. Strubecker, and J. M. Feld have made contributions to this subject.

A turn T_α upon a surface Σ is a lineal element transformation whereby each lineal element is rotated about its point through a constant angle α. A slide S_k is a lineal element transformation whereby to each lineal element E of the surface Σ, there corresponds a lineal element E^* of Σ, such that E and E^* are tangent to the same geodesic of Σ, and the geodesic distance between the points of E and E^* is the constant k. The turns and slides upon a surface Σ form a three-parameter group, called the whirl group W_3, if and only if Σ is of constant Gaussian curvature. In this case, the whirl group W_3 is isomorphic to the motion group M_3 of Σ. The whirl-motion group G_6 of a surface Σ of constant curvature is the six-parameter group of whirls and motions on Σ. The differential geometry of the whirl-motion group G_6 in the Euclidean plane has been developed by Kasner, De Cicco, and Feld in various papers. In the present paper, Feld gives a comprehensive treatment of the corresponding theory upon a spherical surface Σ. The method of presentation is by the quaternion algebra of Hamilton. In analogy to the corresponding treatment in the plane, turbines and flat fields upon a sphere Σ are defined. The turbine group on Σ consists of all lineal element transformations which carry every turbine into a turbine, and hence every flat field into a flat field. Subgroups of the turbine group are G_6, W_3, M_3. The differential geometry of a series of lineal elements on Σ is studied relative to M_3, W_3, G_6. In the whirl-motion geometry of a series on Σ, the Serret-Frenet formulas and the intrinsic equations are obtained. Finally kinematic interpretations are given. *J. De Cicco* (Chicago, Ill.).

Mishra, R. S. On ruled surfaces. Bull. Calcutta Math. Soc. 43, 67–70 (1951).

L'auteur applique le calcul des vecteurs dualistiques de Study à l'étude des surfaces réglées. Il attache à chaque génératrice d'une surface réglée le repère trirectangle (principal) formé par la génératrice, la normale au point central et la tangente à la surface en ce même point normale aux deux droites précédentes. Il détermine ainsi, pour chacune des surfaces réglées engendrées par les trois arêtes du trièdre principal la "skewness of distribution", notion introduite par Rangachariar [même Bull. 37, 133–136 (1945); ces Rev. 7, 392], puis applique un résultat de Ram Behari [Lucknow Univ. Studies, no. 18 (1946); ces Rev. 9, 378] à la recherche des conditions pour que les surfaces réglées en jeu admettent des quadriques osculatrices équilatères. *P. Vincensini* (Marseille).

Jonas, Hans. Die Bestimmung der Hauptflächen und der Developpablen im pseudosphärischen Strahlensystem und eine Eigenschaft gewisser Evolventenflächen des verbogenen Katenoids. Math. Nachr. 6, 293–302 (1952).

In the consideration of pseudospherical systems, the Bäcklund transformation is useful. It is found that by introducing a second such transformation, an integrating factor can be found for the required differential equation and the integral can then be written in closed form involving only such quadratures as occur in the second Bäcklund transformation. This technique leads to determination of the principal surfaces and developables of a pseudospherical congruence. The major results obtained are the following. 1) The determination of the principal surfaces of a pseudospherical congruence generated by the Bäcklund transformation B_σ reduces to quadratures as soon as a solution of the Riccati equation for the complementary transformation $B_\sigma = 0$ is known. 2) The determination of the developables in the pseudospherical congruence generated by the Bäcklund transformation B_σ requires only quadratures as soon as a solution of the Riccati equation for the opposite Bäcklund transformation $B_{-\sigma}$ is known. *S. B. Jackson.*

Jonas, Hans. Anwendung der Bäcklund-Transformation auf die Flächen, die mit den pseudosphärischen gemeinsame Gausssche Bilder der Krümmungslinien haben. Math. Nachr. 6, 327–342 (1952).

A normal congruence of lines consists of the normals to a surface. If the congruence is also a cyclic congruence, the surface is characterized by the fact that the spherical images of the lines of curvature are the same as for a pseudospherical surface. The behavior of this class of surfaces under transformations holding for pseudospherical surfaces and which are actually transformations of the image sphere forms the content of this paper. The transformations involved are the Ribaucour transformation and the Bäcklund transformation.

S. B. Jackson (College Park, Md.).

Tomonaga, Yasuro. A generalization of Laguerre geometry. II. J. Math. Soc. Japan 3, 310–316 (1951).

This paper continues the author's program of generalizing the differential Laguerre geometry by applying tensor calculus [cf. J. Math. Soc. Japan 2, 253–266 (1951); these Rev. 13, 158]. Groups of holonomy are defined as in Riemannian geometry. The case where the group of holonomy fixes one hypersphere is studied first in considerable detail. Then in

the case where the group of holonomy fixes two hyperspheres, one arbitrary, but the second with coordinates 0 for center, it is shown that R_n contains ∞^1 totally umbilic hypersurfaces with constant mean curvature and with orthogonal trajectories which are geodesics. If a hypersphere which touches a hypersurface is fixed by the group of holonomy along the hypersurface, the hypersurface is called a W-hypersurface. In considering the case where the group of holonomy fixes two arbitrary hyperspheres, the author shows that if these hyperspheres satisfy a certain fairly simple inequality in a domain of R_n, that domain contains ∞^1 W-hypersurfaces, whose orthogonal trajectories are written down. *A. Schwartz* (New York, N. Y.).

Löbell, Frank. Richtungsübertragungen auf einer Fläche. Jber. Deutsch. Math. Verein. **55**, Abt. 1, 89–119 (1952).

The author develops the theory of displacements of a direction over a surface lying in ordinary space. The basic ideas are those of Hessenberg [Jber. Deutsch. Math. Verein. **33**, Abt. 2, 93–95 (1925)], and the principal method used is the kinematic technique developed by the author [Math. Ann. **121**, 427–445 (1950); these Rev. **11**, 686]. The axioms are: (I) Every direction $\mathfrak{e}$ at the initial point of an arc $\mathfrak{c}$ determines uniquely a direction $\mathfrak{e}'$ at the end point of $\mathfrak{c}$; or $\mathfrak{e}\xrightarrow[\mathfrak{c}_1]{}\mathfrak{e}'$. (II) If two arcs $\mathfrak{c}_1$ and $\mathfrak{c}_2$ have the same end point, then $\mathfrak{e}\xrightarrow[\mathfrak{c}_1]{}\mathfrak{e}''$ and $\mathfrak{e}'\xrightarrow[\mathfrak{c}_2]{}\mathfrak{e}''$ imply that $\mathfrak{e}\xrightarrow[\mathfrak{c}_1-\mathfrak{c}_2]{}\mathfrak{e}'$. (III) If $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are two directions at the same point, a displacement along the same arc does not alter the signed angle between them. (IV) If a direction is determined by the vector $\mathfrak{e}$, the length of this vector is unchanged after a displacement. (V) Displacements are continuous and differentiable to a required order.

Let the displacement be given by $\mathfrak{e}(s)$ where $\mathfrak{e}$ is a vector and s is the arc length of the curve. Let the unit normal to the surface be $\mathfrak{n}$. The frame $(\mathfrak{e}, \mathfrak{n}\times\mathfrak{e}, \mathfrak{n})$ then rotates with an angular velocity vector $\bar{\mathfrak{u}}$. The tangential component of $\bar{\mathfrak{u}}$ is $\mathfrak{g}=\mathfrak{n}\times d\mathfrak{n}/ds$ where $\mathfrak{g}$ is called the geodesic curvature vector. It depends only on the surface and the arc and has nothing to do with the displacement. The normal component of $\bar{\mathfrak{u}}$, called Ψ, then determines the displacement. Ψ is a function of the line element $(P, \mathfrak{t})$ such that $\Psi(P, -\mathfrak{t}) = -\Psi(P, \mathfrak{t})$. This function is called the characteristic function of the displacement. The characteristic curves for a displacement (corresponding to geodesics) are those for which $\bar{\mathfrak{u}}$ has only a normal component. The equations for these curves are obtained.

A displacement is "linear" if $\bar{\mathfrak{u}}=\mathrm{A}\mathfrak{t}$ where A is an affinor, not necessarily symmetric. The characteristic function of a linear displacement then has the form $\Psi=\mathfrak{v}\mathfrak{t}$ (scalar product). If $\Psi=0$, the displacement is "geodesic" or "parallel"; such a displacement depends only on the intrinsic metric of the surface. If the displacement is independent of the path, it is called "integrable". Every integrable displacement must be linear. Integrability conditions are obtained in a familiar form. Applications are made to surface theory.

C. B. Allendoerfer (Seattle, Wash.).

Bakel'man, I. Ya. Smooth surfaces of bounded curvature. Doklady Akad. Nauk SSSR (N.S.) **82**, 501–504 (1952). (Russian)

Let $z=f(x, y)$ be defined in a square Q and have continuous first derivatives. Let P be a polygonal region in Q, and $\Delta_1, \cdots, \Delta_k$ the triangles of a triangulation T of P. Denote by $\varphi_{i1}, \varphi_{i2}, \varphi_{i3}$ the angles between the normals to the tangent planes of F: $z=f(x, y)$ at the points projected into the vertices of Δ_i. Put $\mu(\Delta_i)=\varphi^2_{i1}+\varphi^2_{i2}+\varphi^2_{i3}$, $\mu(T)=\sum\mu(\Delta_i)$. For a fixed constant $c>0$ consider all triangulations T_c for which $\sum\delta_i^2\leqq cA(P)$ where δ_i is the diameter of Δ_i and $A(P)$ the arc of P. Put $\mu_c(P)=\sup_{T_c}\mu(T_c)$ and $\mu(P)=\lim_{c\to\infty}\mu_c(P)/c$. If G is an open subset of F, then the μ-curvature of G is the number $\mu_0(G)=\sup\sum\mu(P_i)$, where $(P_1, \cdots, P_k)$ traverses all sets of non-overlapping polygons in the projection G' of G in Q. If $\sup_{G\subset F}\mu_0(G)<\infty$ then F is said to have bounded μ-curvature. (The Russian word in the title is a variant of the usual word for curvature, thus a term like μ-curvature is avoided.)

Every surface of bounded μ-curvature is a surface of bounded curvature in the sense of A. D. Alexandrov. A surface F of class C' has bounded μ-curvature, if and only if it can be approximated together with its derivatives by a sequence of regular surfaces F_n for which the integrals $\int\int[\kappa^2_{n1}(x)+\kappa^2_{n2}(x)]dS_x$ are uniformly bounded; here $\kappa_{ni}(x)$ are the principal curvatures of F_n at x and dS_x is the arc element of F_n at x. For any surface of bounded μ-curvature the curvature in the sense of Alexandrov equals the arc of the spherical image. Characterizations of surfaces of negative and positive curvature in terms of μ-curvature are given to show the usefulness of the concept.

H. Busemann (Auckland).

Grotemeyer, K. P. Zur eindeutigen Bestimmung von Flächen durch die erste Fundamentalform. Math. Z. **55**, 253–268 (1952).

Given two isometric three-times continuously differentiable surfaces $\mathfrak{F}=\mathfrak{x}=\mathfrak{x}(u, v)$ and $\mathfrak{F}^*=\mathfrak{x}^*=\mathfrak{x}^*(u, v)$. Notations: $ds^2=Edu^2+2Fdudv+Gdv^2$ is the line element of $\mathfrak{F}$ and $\mathfrak{F}^*$ $[W=\sqrt{(EG-F^2)}>0]$; K=Gaussian curvature;

$$\mathfrak{N}=\mathfrak{x}_u\times\mathfrak{x}_v/W \quad\text{and}\quad \mathfrak{N}^*=\mathfrak{x}_u^*\times\mathfrak{x}_v^*/W$$

are their normal vectors;

$$Ldu^2+2Mdudv+Ndv^2 \quad\text{and}\quad L^*du^2+2M^*dudv+N^*dv^2$$

are their second fundamental forms. Applying Herglotz' method [Abh. Math. Sem. Hansischen Univ. **15**, 127–129 (1943); these Rev. **7**, 322], the surface integral

$$(1)\qquad \int_{\mathfrak{F}}\begin{vmatrix} L^*-L & M^*-M \\ M^*-M & N^*-N \end{vmatrix}(\mathfrak{N}\mathfrak{x}+\mathfrak{N}^*\mathfrak{x}^*)W^{-2}do$$

is expressed by a contour integral. If $\mathfrak{F}$ and $\mathfrak{F}^*$ are closed surfaces, (1) will vanish and a discussion of the sign of its integrand yields the uniqueness theorem for convex surfaces with $K>0$. The author's original contribution consists in three uniqueness theorems for surfaces with holes; K is supposed to be positive in the interiors of $\mathfrak{F}$ and $\mathfrak{F}^*$; their boundaries correspond to each other and satisfy various additional conditions. It may be sufficient to quote those for Theorem 6: Each boundary curve lies in a plane which is the tangent plane of the surface along the entire curve [cf. Stoker, Comm. Pure Appl. Math. **3**, 231–257 (1950); these Rev. **12**, 631]. The proofs are based on the discussion of (1). *P. Scherk* (Los Angeles, Calif.).

Dorfman, A. G. Transformation of the equation of deformation. Uspehi Matem. Nauk (N.S.) **6**, no. 6(46), 165–166 (1951). (Russian)

Dorfman, A. G. The deformation of a surface with a flat point. Uspehi Matem. Nauk (N.S.) **6**, no. 6(46), 167–173 (1951). (Russian)

These papers form an addition to N. V. Efimov's report on qualitative questions in the theory of deformation of surfaces in the small [Trudy Mat. Inst. Steklov. **30** (1949);

these Rev. 12, 531]. There the concept of relative rigidity of a given order is defined, but only flat points of rather high order are considered. The second of the present two papers establishes the following theorem on flat points of the lowest order: Let the surface S have the form

$$x=u,\quad y=v,\quad z=f^3(u,v)+\cdots+f^{(3+h)}(u,v)+R(u,v)$$

where $f^n(u,v)=\sum_{k=3}^{n}\binom{n}{k}a_k{}^n u^{n-k}v^k$ and $R(u,v)$ is the remainder. Put $p_{ij}=a_i{}^3a^3{}_{j+1}-a^3{}_{i+1}a_j{}^3$. If $4p_{01}p_{12}-p^2{}_{02}\neq 0$ and

$$p^2{}_{12}a_0{}^4-2p_{12}p_{02}a_1{}^4+(p^2{}_{02}+2p_{01}p_{12})a_2{}^4 \\ -2p_{01}p_{02}a_3{}^4+p^2{}_{01}a_4{}^4\neq 0,$$

then S is relatively rigid of at least the first order. The main tool in the proof is the following comparatively simple form of the deformation equation derived in the first paper: If one surface $z=f(u,v)$ with a given line element is known, then any other surface $z(u,v)$ with the same line element satisfies the equation

$$(z_{uu}z_{vv}-z^2{}_{uv})(f_u{}^2+f_v{}^2)-(f_{uu}z_{vv}-2f_{uv}z_{uv}+f_{vv}z_{uu})(f_uz_u+f_vz_v) \\ +(f_{uu}f_{vv}-f^2{}_{uv})(z_u{}^2+z_v{}^2)=(f_{uu}f_{vv}-f^2{}_{uv})-(z_{uu}z_{vv}-z^2{}_{uv}).$$

H. Busemann (Auckland).

Matsumoto, Makoto. Conformally flat Riemann spaces of class one. J. Math. Soc. Japan 3, 306–309 (1951).

The author obtains an algebraic characterization that a conformally flat Riemann space be of class one. The characterization states that for $n\geqq 4$ a positive definite conformally flat space (not of constant curvature) is of class one if a certain matrix is of rank $\geqq 2$ and if certain equalities are satisfied. The matrix and these equalities involve only g_{ij} and R_{ijkl}. A new proof is given of the result that the class of a conformally flat space is at most two.

C. B. Allendoerfer (Seattle, Wash.).

Kurita, Minoru. Characterization of certain Riemann spaces by development. Nagoya Math. J. 3, 81–90 (1951).

Here is presented a geometrical characterization of Riemann spaces with metric of the form

$$ds^2=a(x^\gamma)^2g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta+g_{\lambda\mu}(x^\nu)dx^\lambda dx^\mu$$

where $i,j=1,2,\cdots,n;\ \alpha,\beta,\gamma=1,2,\cdots,k;\ \lambda,\mu=k+1,\cdots,n$. They may be considered a generalization of spaces studied by K. Yano [Proc. Imp. Acad. Tokyo 16, 83–86 (1940); these Rev. 1, 273]. Cartan's exterior calculus is used to good effect. Necessary and sufficient conditions for the above form of linear element are obtained as relatively simple geometric properties of the point displacements of the tangent spaces defined by the linear connection associated with the metric.

J. L. Vanderslice (Silver Spring, Md.).

Adati, Tyuzi. On Riemannian spaces admitting a family of totally umbilical hypersurfaces. I. Proc. Japan Acad. 27, 1–6 (1951).

Various results concerning a Riemann space V_n admitting a family of totally umbilical hypersurfaces are obtained. The cases in which V_n is an Einstein space or the hypersurfaces are conformally flat are studied. *A. Fialkow.*

Adati, Tyuzi. On Riemannian spaces admitting a family of totally umbilical hypersurfaces. II. Proc. Japan Acad. 27, 49–54 (1951).

This paper is a continuation of the paper reviewed above dealing principally with torse-forming vector fields. A typical result is the following: When a conformally flat space V_n $(n>3)$ admits a torse-forming vector field $\partial\sigma/\partial x_i$, the hypersurfaces $\sigma(x_i)=$constant are of constant Riemann curvature. *A. Fialkow* (Brooklyn, N. Y.).

Ōtsuki, Tominosuke. On the spaces with normal projective connexions and some imbedding problem of Riemannian spaces. Math. J. Okayama Univ. 1, 69–98 (1952).

The major part of the paper is concerned with the analytical aspects of the following two geometrical problems: 1) study of Riemannian spaces such that the group of holonomy of the normal projective connection arising from the geodesics fixes a hyperquadric; 2) study of Riemannian space which can be imbedded as a hypersurface F in an Einstein space whose normal projective connection has a group of holonomy leaving invariant a hyperquadric with F as its image. In the first case the space is projective in the neighborhood of a generic point to an Einstein space with non-vanishing curvature. *S. Chern* (Chicago, Ill.).

Takeno, Hyôitirô. Theory of the spherically symmetric space-times. I. Characteristic system. J. Math. Soc. Japan 3, 317–329 (1951).

The author defines a space-time of signature $(---+)$ to be spherically symmetric if there exist mutually orthogonal unit vectors α_i (space-like) and β_i (time-like) in terms of which the curvature tensor is given by

$$K_{ijlm}=-\rho_1\alpha_{[i}\alpha_{[l}\beta_{j]}\beta_{m]}-\rho_2g_{[i[l}\alpha_{j]}\alpha_{m]} \\ +\rho_3g_{[i[l}\beta_{j]}\beta_{m]}+\rho_4g_{[i[l}g_{j]m]},$$

the ρ's being scalars, while α_i and β_i satisfy

$$\nabla_i\alpha_j=\sigma\alpha_i\beta_j+\kappa(g_{ij}+\alpha_i\alpha_j-\beta_i\beta_j)+\sigma'\beta_i\beta_j,$$
$$\nabla_i\beta_j=\sigma'\beta_i\alpha_j+\kappa'(g_{ij}+\alpha_i\alpha_j-\beta_i\beta_j)+\sigma\alpha_i\alpha_j,$$

where σ, σ', κ, κ' are scalars. Two further subsidiary conditions, which need not be quoted here, are also imposed. The quantities $(\alpha_i,\beta_i,\rho_a,\sigma,\sigma',\kappa,\kappa')$ constitute the "characteristic system" referred to in the title of the paper. The vectors α^i, β^i turn out to be Ricci principal directions. By a suitable choice of coordinates, ds^2 is reducible to the standard form

$$ds^2=-A(r,t)dr^2-B(r,t)(d\theta^2+\sin^2\theta d\phi^2)+C(r,t)dt^2.$$

The author discusses transformations of characteristic systems, and numerous special cases, and obtains, incidentally, a rule for determining whether a spherically symmetric space-time given in terms of general coordinates is reducible to static form. *H. S. Ruse* (Princeton, N. J.).

Patterson, E. M., and Walker, A. G. Riemann extensions. Quart. J. Math., Oxford Ser. (2) 3, 19–28 (1952).

La notion d'extension riemannienne a été dégagée par les auteurs à partir de leurs études sur les variétés riemanniennes qui admettent des champs parallèles de plans partiellement nuls. Le cas intéressant ici est celui d'un espace riemannien R^{2n} qui admet un champ parallèle π de n-plans nuls, engendré par un tenseur symétrique récurrent $T_{\alpha\beta}$ $(\alpha,\beta=1,\cdots,2n)$. La première partie du papier est consacrée à l'étude locale détaillée de ce cas, R^{2n} admettant alors une métrique de la forme

$$(*)\quad ds^2=g_{ij}(x,\xi)dx^idx^j+2dx^id\xi_i \qquad (i,j \text{ et tout indice latin }=1,\cdots,n).$$

De cette étude, il résulte que, quand la courbure riemannienne scalaire R de R^{2n} est une constante non nulle, la donnée de R^{2n} admettant le champ π détermine d'une manière unique un espace conforme CR^n et, dans cet espace, un tenseur a_{ij} invariant par transformation conforme et relié

d'une manière simple au tenseur d'Einstein de R^{2n}. Dans le cas où $R=0$, on peut associer à R^{2n} d'une manière unique un espace de Weyl W^n. Dans une seconde partie, les auteurs étudient le problème inverse, c'est-à-dire la construction à partir d'espaces conformes ou à connexion affine à n dimensions d'extensions riemanniennes locales à $2n$ dimensions. Dans le cas conforme, la donnée du tenseur fondamental h_{ij} de CR^n, celle d'un scalaire constant non nul, et d'un champ de tenseurs symétriques a_{ij} dans CR^n permettent la construction d'un R^{2n}, extension riemannienne de CR^n, telle que les éléments précédents se déduisent de ceux de R^{2n} conformément aux résultats de la première partie ($R\neq 0$). En particulier on peut toujours choisir par extension riemannienne un espace d'Einstein en prenant $a_{ij}=0$. Dans le cas d'un espace A^n à connexion symétrique affine $L^p{}_{ij}$, une extension riemannienne R^{2n} (A^n, c_{ij}) est donnée par l'espace de coordonnées (x^i, ξ_i), de métrique (*), avec

$$g_{ij}=-2L^p{}_{ij}\xi_p+c_{ij}(x).$$

La courbure riemannienne scalaire est alors nulle. Dans une troisième partie, les auteurs donnent la notion globale d'extension riemannienne. Etant donné une variété différentiable M^n, ils introduisent l'espace fibré M^{2n} des vecteurs covariants (x^i, ξ_i) aux différents points (x^i) de M^n. Si M^n admet une structure conforme ou une connexion affine symétrique, les considérations précédentes peuvent être appliquées à chaque voisinage de coordonnées et permettent de douer M^{2n} de structures globalement riemanniennes privilégiées; $\bar{M}^{2n}$ pourvue d'une telle structure sera dite une extension riemannienne globale de M^n. *A. Lichnerowicz.*

Patterson, E. M. Simply harmonic Riemann extensions. J. London Math. Soc. 27, 102–107 (1952).

Etant donné un espace A^n à connexion affine symétrique et un tenseur symétrique c_{ij} sur A^n, l'auteur se propose de rechercher à quelles conditions l'extension riemannienne R^{2n} (A^n, c_{ij}) [voir l'analyse précédente] est simplement harmonique. Ces conditions, de forme assez compliquée, sont explicitement obtenues par une méthode dont le principe est dû à A. G. Walker [Proc. Edinburgh Math. Soc. (2) 7, 16–26 (1942); ces Rev. 4, 171]. L'auteur en déduit alors le théorème suivant: une extension riemannienne R^{2n} (R^n, c_{ij}) d'un espace riemannien R^n est simplement harmonique si et seulement si R^n est simplement harmonique. Ce théorème englobe un résultat précédent de l'auteur [J. London Math. Soc. 26, 238–240 (1951); ces Rev. 12, 858] sur les R^{2n} admettant un champ strictement parallèle de n-plans nuls (cas où R^n est un espace euclidien). L'auteur propose d'appeler espaces à connexion affine simplement harmoniques les A^n satisfaisant aux conditions trouvées et qui engendrent les extensions riemanniennes simplement harmoniques. Il montre que pour qu'un A^2 soit simplement harmonique, il faut et il suffit que ce soit un espace de Weyl dont le tenseur de courbure contracté soit antisymétrique; des considérations précédentes, il déduit de nouveaux exemples de R^n simplement harmoniques.

A. Lichnerowicz (Paris).

Borel, Armand, et Lichnerowicz, André. Groupes d'holonomie des variétés riemanniennes. C. R. Acad. Sci. Paris 234, 1835–1837 (1952).

Using the method of development of an arc in a space with an infinitesimal connexion [Ehresmann, Colloque de topologie, Bruxelles, 1950, Thone, Liège, 1951, pp. 29–55; these Rev. 13, 159] certain groups of holonomy are defined for a Riemannian manifold. Some properties of these groups are obtained, and the condition for the existence of a Kähler structure is expressed in terms of the groups.

W. V. D. Hodge (Cambridge, England).

Borel, Armand, et Lichnerowicz, André. Espaces riemanniens et hermitiens symétriques. C. R. Acad. Sci. Paris 234, 2332–2334 (1952).

This paper announces some general theorems on symmetric Riemannian spaces, i.e., Riemannian spaces V_m such that, for each point x of V_m, there exists an involutory isometric transformation of V_m into itself having x as an isolated fixed point. These spaces can be obtained as follows: let G be a Lie group, S an involutory automorphism of G, and H a compact subgroup of G such that (a) S reduces to identity on H, (b) H contains the largest connected subgroup of G on which S reduces to identity; then $V_m=G/H$ is a symmetric Riemannian space. In the case in which G is semi-simple, necessary and sufficient conditions are obtained for V_m to have a complex structure which is invariant under the group G of symmetries of V_m. When the conditions are satisfied, V_m is necessarily Kählerian. A list of the symmetric spaces of this type is given. *W. V. D. Hodge.*

Sasaki, Shigeo. On a theorem concerning the homological structure and the holonomy groups of closed orientable symmetric spaces. Proc. Japan Acad. 27, 81–85 (1951).

Le but de l'auteur est d'établir le théorème suivant: Soit $M_n=G/H$ un espace symétrique supposé compact et orientable. La donnée sur M_n d'une métrique invariante par G doue M_n d'une structure d'espace riemannien symétrique. Alors toute forme harmonique de M_n est à dérivée covariante nulle. La démonstration proposée ne paraît pas satisfaisante au rapporteur: le pas principal de cette démonstration consiste, selon l'auteur, à établir que le groupe linéaire d'isotropie coïncide avec le groupe d'holonomie homogène. Il n'est pas précisé s'il s'agit des groupes eux-mêmes ou des composantes connexes de e de ces groupes et la preuve donnée n'est susceptible d'atteindre que ces dernières. De plus le cas où M_n est localement euclidien—ou bien admet une métrique dont une composante est localement euclidienne—n'est pas écarté, alors qu'il met visiblement en défaut l'assertion précédente. Une étude plus précise d'A. Borel et du rapporteur [voir les deux analyses précédentes], basée sur l'approximation des lacets par des polygones géodésiques, montre que le groupe d'holonomie homogène ψ_x $(x \in M_n)$ est seulement en général sous-groupe invariant du groupe linéaire d'isotropie $\bar{H}$. Ce résultat suffit d'ailleurs à établir le théorème cherché par l'auteur. De plus, si G est semi-simple—en particulier si l'espace M_n est irréductible—les composantes connexes de e de ψ_x et $\bar{H}$ coïncident.

A. Lichnerowicz (Paris).

Duff, G. F. D. Differential forms in manifolds with boundary. Ann. of Math. (2) 56, 115–127 (1952).

De Rham's two theorems [J. Math. Pures Appl. (9) 10, 115–200 (1931)] deal with exterior forms on a closed manifold. This paper extends de Rham's results to orientable manifolds M with regular boundary B. The method depends on embedding M in a closed orientable manifold F (dim F=dim B). The generalisations give rise to several cases according as the forms on M vanish on B, or are considered in respect of their periods on the absolute cycles of M and their boundary values on B, or in respect of their periods on the relative cycles of M mod B and their boundary values. *W. V. D. Hodge* (Cambridge, England).

Duff, G. F. D., and Spencer, D. C. Harmonic tensors on Riemannian manifolds with boundary. Ann. of Math. (2) 56, 128–156 (1952).

This paper deals with the problem of the existence of harmonic forms on an orientable Riemannian manifold M with regular boundary B which have assigned boundary values and assigned periods on the absolute cycles of M or the relative cycles mod B. The paper makes use of the results of the paper by Duff reviewed above, and the problems solved correspond to the various cases considered in that paper. Different forms of the harmonic condition are considered, the problem being solved for forms f which satisfy one of the following conditions in M:

(i) $(d\delta+\delta d)f=0;$

(ii) $d\delta f=0 \quad (\text{or } \delta d f=0);$

(iii) $df=0, \quad \delta f=0.$

W. V. D. Hodge (Cambridge, England).

Gaffney, Matthew P. The harmonic operator for exterior differential forms. Proc. Nat. Acad. Sci. U. S. A. 37, 48–50 (1951).

Let $\Delta=d\delta+\delta d$ be the harmonic operator defined for exterior differential forms on a Riemannian manifold of class C^3. The operator Δ can be regarded as a linear transformation (with a properly selected domain) on the Hilbert space consisting of all square integrable differential forms on the manifold. From this point of view, one of the important questions is whether or not the closure $\bar{\Delta}$ of Δ is self-adjoint. The domain of d (or δ) is taken to be the set of all forms α of class C^1 such that both α and $d\alpha$ (or $\delta\alpha$) are square integrable. The domain of $\Delta=d\delta+\delta d$ is therefore the set of those α such that α, $d\alpha$, $\delta\alpha$ are all in the domains of d and δ. Now the manifold is said to have "negligible boundary" if $(d\alpha, \beta)=(\alpha, \delta\beta)$ holds for all α, β belonging to the domains of d, δ, respectively. The main result of this paper is that for manifolds with negligible boundary the closure $\bar{\Delta}$ is self-adjoint. An outline of a proof of this result based on the Friedrichs mollifier is given. It is also stated that a manifold has negligible boundary if there exists a function s with the following properties: Outside some fixed compact set, i) s is of class C^1, ii) the set $\{p \mid s(p)\leqq t\}$ is compact for each $t\geqq t_0$, and iii) the gradient of s is never zero and the magnitude of the gradient is bounded. *K. Kodaira* (Princeton, N. J.).

Reeb, G. Sur les éléments de contact linéaires du second ordre attachés à un système différentiel. J. Reine Angew. Math. 189, 186–189 (1951).

"Soit V_n une variété compacte orientable à n dimensions de classes G_4, douée d'une structure d'espace de Riemann (ou de Finsler). Parmi les éléments de contact linéaires du second ordre attachés à V_n on peut distinguer ceux qui sont géodésiques pour la structure envisagée. En utilisant certaines propriétés de l'invariant intégral de E. Cartan, nous montrerons que parmi les éléments de contact du second ordre attachés aux lignes intégrales d'une équation de Pfaff $\omega=0$, définie dans V_n, il y a certainement des éléments de contact géodésiques." (From the author's introduction.)

S. S. Chern (Chicago, Ill.).

Rund, Hanno. Zur Begründung der Differentialgeometrie der Minkowskischen Räume. Arch. Math. 3, 60–69 (1952).

One of the first notions for the geometry in a Minkowski space is that of angle. The author adopts one which essentially generalizes Bliss's angle in the two-dimensional case. From this various notions on the differential geometry of curves and hypersurfaces are developed. Topics include: curvature of a curve, Dupin indicatrix, principal directions, analogues of formulas of Rodrigues, Meusnier's theorem, etc. In a final remark the author promises to develop in a future paper a theory of parallelism of Finsler spaces, which differs essentially from those of E. Cartan and Berwald.

S. Chern (Chicago, Ill.).

Nijenhuis, A. An application of anholonomic coordinates. Math. Centrum Amsterdam. Rapport ZW-1951-017, 6 pp. (1951). (Dutch)

The anholonomic coordinates in an X_n and the object of anholonomity $\Omega^h{}_{ji}$ are defined. As an example it is first proved that the measuring vectors $\underset{i}{e}^h$, $\underset{j}{e}^h$ span a system of ∞^{n-2} X_2's if and only if $\Omega^h{}_{ji}=0$ for h, i, j not equal. Using the eigen-vectors of an affinor $h_\lambda^{\cdot\kappa}$ with all different eigenvalues $\underset{i}{\lambda}$ as measuring vectors of an anholonomic system, the invariant $H_{\mu\lambda}^{\cdot\cdot\kappa}$ [cf. Nijenhuis, Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 200–212 (1951); these Rev. 13, 281], whose vanishing is necessary and sufficient for the $\overset{h}{e}_\lambda$ to be X_{n-1}-building, can be put in the very simple form

$$H_{ji}^{\cdot\cdot h}=(\underset{j}{\lambda}-\underset{i}{\lambda})(H_i{}^h\partial_j\underset{i}{\lambda}-A_j{}^h\partial_i\underset{j}{\lambda})-2(\underset{h}{\lambda}-\underset{i}{\lambda})(\underset{h}{\lambda}-\underset{j}{\lambda})\Omega_{ji}^{\cdot\cdot h}.$$

J. A. Schouten (Epe).

Goto, Ken-iti. On the so-called pseudo spinors. Progress Theoret. Physics 6, 990–993 (1951).

The author discusses the behavior of four-component simple spinors under transformations corresponding to four types of Lorentz transformations: Proper ones which send the future into the future, improper ones which send the future into the future, proper ones which interchange the future and the past and improper ones which interchange the future and the past. He defines four classes of simple spinors and shows that one pair of classes is equivalent (under linear transformations) to another pair. These results and the antilinear relation between the remaining two unequivalent classes of spinors may be readily derived from the geometric interpretation of spinors propounded by Veblen and his coworkers.

A. H. Taub (Urbana, Ill.).

NUMERICAL AND GRAPHICAL METHODS

✓ ***Milne-Thomson, L. M. Jacobian elliptic function tables. A guide to practical computation with elliptic functions and integrals together with tables of sn u, cn u, dn u, $Z(u)$.** Dover Publications, Inc., New York, N. Y., 1950. xi+123 pp. $2.45.

The chief problem in using elliptic functions has always been the lack of suitable numerical tables. Therefore the collection of tables, compiled by the author is an excellent manual, which supplies a long-felt want. It contains the following.

1) Five-figure tables of the functions

sn $(u|m)$, cn $(u|m)$ and dn $(u|m)$ $(m=k^2)$ for

$u=$	0(.01)2	for $m=$	0(.1)1
	2(.01)2.50		0.6(.1)1
	2.50(.01)2.75		0.9, 1.
	2.75(.01)3		1.

2) Eight-figure tables of the complete elliptic integrals K, K', E, E' and the nome q as functions of m, $m=0(.01)1$.

3) Seven-figure table of the Jacobian Zeta-function $Z(u)$ for

$u=$ 0(.01)2	$m=0.1(.1)1$
2(.01)2.50	0.7(.1)1
2.5(.01)3	0.9, 1.

A comprehensive collection of formulae is included, and many numerical examples are given in order to illustrate the use of these tables. The formulae have been carefully chosen with a view to facilitating calculations which may present themselves. Formulae special to particular branches of knowledge are not included, with the exception of some conformal transformations which cover ground common to several sciences. *S. C. van Veen* (Delft).

✓***Stanley, J. P., and Wilkes, M. V. Table of the reciprocal of the gamma function for complex argument.** Computation Centre, University of Toronto, Toronto, Ont., 1950. Unpaged.

The table gives 6 decimal values of the function

$$1/\Gamma(Z)=U+iV,$$

where $Z=X+iY$ and

$$X=-0.50(.01)+0.50, \quad Y=0.00(.01)1.00.$$

The function is calculated from the series $1/\Gamma(Z)=\sum_{n=1}^{\infty}C_nZ^n$ $(C_1=1)$. The coefficients, first computed by L. Bourget, are given by H. T. Davis, "Tables of the higher mathematical functions", vol. 1, p. 285 [Principia Press, Bloomington, Indiana, 1933]. Twenty-one terms of the series were used. The entire calculation was performed on the electronic digital computer EDSAC at the Mathematical Laboratory, Cambridge, England. Ten decimals were carried throughout the computation in order to minimize rounding-off errors. The table was checked by differencing in both directions on a National Accounting Machine. Moreover, the table has been checked against all previous (less extensive) tabulations. The maximum error in the table is estimated by the authors to be not greater than 0.7 units in the sixth decimal place, made up as follows:

Final rounding-off error:	0.50 units
Truncation error:	0.04 units
Rounding-off errors accumulated during calculations:	0.16 units.

A table of errata noted by C. M. Munford of the University Mathematical Laboratory, Cambridge and A. van Wijngaarden of the Mathematical Centre of Amsterdam is added to the table. Values lying outside the range of this table may be obtained through the use of one or both of the formulae

$$\Gamma(Z+1)=Z\Gamma(Z), \quad \Gamma(2Z)=2^{2Z-1}\pi^{-\frac{1}{2}}\Gamma(Z)\Gamma(Z+\tfrac{1}{2}).$$

S. C. van Veen (Delft).

✓***Snow, Chester. Hypergeometric and Legendre functions with applications to integral equations of potential theory.** National Bureau of Standards Applied Mathematics Series, No. 19. U. S. Government Printing Office, Washington, D. C., 1952. xi+427 pp. $3.25.

For a review of the first edition see these Rev. 4, 197. In this edition known errors have been corrected, chapter VIII on integral representations and section 7 of chapter X, on annular coordinates, have been rewritten, and a new chapter XI, Integral representations with integrals of confluent hypergeometric functions, has been added.

✓***Tables of Coulomb wave functions. Vol. I.** National Bureau of Standards Applied Mathematics Series, No. 17. U. S. Government Printing Office, Washington, D. C., 1952. xxvii+141 pp. $2.00.

The "regular Coulomb wave function" $F_L(\eta,\rho)$ is that solution of the differential equation

$$\frac{d^2y}{d\rho^2}+\left[1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right]y=0$$

which is regular at the origin and is so normalized that

$$F_L(\eta,\rho)\sim\sin(\rho-\eta\ln(2\rho)-\tfrac{1}{2}L\pi+\sigma_L) \quad \text{as} \quad \rho\to\infty,$$

with $\sigma_L=\arg\Gamma(L+1+i\eta)$. This function is written in the form

$$F_L(\eta,\rho)=C_L(\eta)\rho^{L+1}\Phi_L(\eta,\rho)$$

where

$$C_0(\eta)=[2\pi\eta/(e^{2\pi\eta}-1)]^{1/2},$$
$$C_L(\eta)=(L^2+\eta^2)^{1/2}C_{L-1}/[L(2L+1)],$$

and Φ is a power series in ρ.

Table I (pp. 1–111) of the present work gives 7D values of $\Phi_L(\eta,\rho)$ and of as many reduced derivatives $(k!)^{-1}d^k\Phi_L(\eta,\rho)/d\eta^k$ as are needed for interpolation, for $\eta=-5(1)5$, $L=0(1)5$, 10, 11, 20, 21, and $\rho=0(.2)5$. Table II (pp. 113–127) gives 10D values of the real part of $\Gamma'(1+i\eta)/\Gamma(1+i\eta)$, with modified second central differences, for $\eta=0(.005)2(.01)6(.02)10(.1)20(.2)60(.5)110$. Table III gives 8D values of σ_0, with modified second central differences, for $\eta=0(.01)1(.02)3(.05)10(.2)20(.4)30(.5)85$. Table IV (pp. 137–141) gives 8S values of $C_0(\eta)$ for $\eta=0(.01)5(.05)10$.

The Foreword by G. Breit contains an account of the applications of Coulomb wave functions. [This account is a slightly enlarged version of a portion of a paper by Bloch et al., Rev. Modern Physics 23, 147–182 (1951); these Rev. 13, 234.] The Introduction by M. Abramowitz contains the definitions of Coulomb wave functions and a collection of formulas, Bessel function expansions, asymptotic expansions, a description of the method of computation, detailed instructions for interpolation and for the use of the recurrence relations, and a brief section on the quantities tabulated in Tables II, III, IV. Both the Foreword and the Introduction contain numerous references but there is no account of other available tables of Coulomb wave functions. The present tables are wider in scope than any other accessible tables, yet the tables by Bloch and others referred to above are not entirely superseded by the newer book. The tables should prove very useful in computations in nuclear physics. *A. Erdélyi* (Pasadena, Calif.).

✓***A guide to tables of the normal probability integral.** National Bureau of Standards Applied Mathematics Series, No. 21. U. S. Government Printing Office, Washington, D. C., 1952. iv+16 pp. $.15.

Mulholland, H. P. On the distribution of a convex even function of several independent rounding-off errors. Proc. Amer. Math. Soc. 3, 310–321 (1952).

To describe the principal results of this paper let $V_k(t)$ be the k-fold convolutions of $V_1(t)$, the uniform distribution on the interval $(-\frac{1}{2},+\frac{1}{2})$. Let b be a positive scale factor and let φ be a convex even function of M-dimensional vectors with $\varphi(0)=0$. If $h_m(m=1,2,\cdots,M)$ are independent

random variables, each having $V_{l_m}(x/b)$, $l_m \leq l$, as their cumulative distribution functions and if a_m are independent random variables, each having the normal cumulative distribution function $\Phi(x/\sigma)$, then the probability that $\varphi(h)>t$ does not exceed the probability that $\varphi(a)>t$, provided that σ is not less than $\rho_l \cdot b(l/12)^{1/2}$ where ρ_l is the order of l. This is a strengthened form of a lemma used by von Neumann and Goldstine [Proc. Amer. Math. Soc. 2, 188–202 (1951); these Rev. 12, 861]. It is proved with the help of a more general theorem. The distribution $F(x)$ is said to be more peaked about 0 than that of $G(x)$ when

$$F(x)-F(-x)=G(x)-G(-x).$$

This concept is due to Birnbaum [Ann. Math. Statistics 19, 76–81 (1948); these Rev. 9, 452]. With the help of this definition the author establishes his basic result. He considers $u_m, v_m (m=1, 2, \cdots, M)$ as independent random variables with unimodal absolutely continuous distributions symmetrical about 0. Then $\varphi(u)$ is more peaked about 0 than that $\varphi(v)$ for φ as above if and only if the distribution of u_m is more peaked about 0 than that of v_m. The paper concludes with an application of his strengthened result to the problem treated by von Neumann and Goldstine mentioned above. This analysis yields slightly improved numerical estimates. *H. H. Goldstine* (Princeton, N. J.).

Martin, D. G. E. Numerical evaluation of the Fermi beta-distribution function. Physical Rev. (2) 81, 280–281 (1951).

The Fermi beta-distribution function

$$f(z, \eta)=\eta^{2+2s} e^{\pi y}|\Gamma(1+s+iy)|^2, \quad z=1+s+iy,$$

is computed by means of the Taylor-expansion

$$\log|\Gamma(1+x+iy)|=\log|\Gamma(m+x+iy)|$$
$$-\tfrac{1}{2}\sum_{k=0}^{m-1}\log\{(k+x)^2+y^2\}+\log\Gamma(m+x)$$
$$+\sum_{k=1}^{\infty}(-1)^k\frac{y^{2k}}{2k!}\psi^{(2k-1)}(m+x).$$

This series converges rapidly if m is large enough, even if y is large $(x>0)$. *S. C. van Veen* (Delft).

Salzer, H. E. Formulas for calculating the error function of a complex variable. Math. Tables and Other Aids to Computation 5, 67–70 (1951).

By means of the well-known expansion

$$\sum_{n=-\infty}^{+\infty}\exp(-(u+na)^2)$$
$$=\pi^{\frac{1}{2}}a^{-1}\sum_{n=-\infty}^{+\infty}\exp(-n^2\pi^2a^{-2})\cos(2n\pi u/a)$$

the author derives the approximate equality

$$e^{u^2}\sim\pi^{-\frac{1}{2}}\left(\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-n^2/4}\cosh nu\right)$$

with a relative error $<2\cdot 10^{-17}$. The integration results in

$$\int_0^Y e^{u^2}du\sim\pi^{-\frac{1}{2}}\left(\tfrac{1}{2}Y+\sum_{n=1}^{\infty}n^{-1}\exp(-n^2/4)\sinh nY\right).$$

For $Z=X+iY$ several approximate expansions are obtained by contour-integration, e.g.,

$$e^{Z^2}\int_0^Z e^{-u^2}du=e^{(X+iY)^2}\int_0^X e^{-u^2}du+2i\sin XYe^{iXY}\pi^{-\frac{1}{2}}e^{-Y^2}$$
$$+2Xe^{2iXY}\sum_{n=1}^{\infty}\frac{e^{-n^2/4}}{n^2+4X^2}$$
$$+\sum_{n=1}^{\infty}\frac{e^{-n^2/4}}{n^2+4X^2}(-2X\cosh nY+in\sinh nY).$$

S. C. van Veen (Delft).

McWeeny, R., and Coulson, C. A. The computation of wave functions in momentum space. I. The helium atom. Proc. Phys. Soc. Sect. A. 62, 509–518 (1949).

The Schrödinger equation for a system of $N-1$ electrons and one nucleus (the Nth particle) is formulated in momentum space. Before attempting the solution, the authors impose the restriction that the total momentum vanish, and also neglect the terms in m/M, which correspond to the motion of the nucleus. For the approximate solution of the simplified integral equation, the iteration method of Svartholm [Dissertation, Lund, 1945] is used, beginning with a suitably chosen product function. The momentum distribution function $P(p)$ to be compared with experimental results is obtained by integrating the absolute square of the antisymmetric wave function over the momenta of all electrons but one. The method is applied with considerable success in the case of the helium atom $(N-1=2)$. The authors state that for more complicated atoms the approximate calculation of wave functions does not present any extraordinary difficulties, but that the subsequent integrations required to obtain $P(p)$ are not feasible.

W. H. Furry (Cambridge, Mass.).

McWeeny, R. The computation of wave functions in momentum space. II. The hydrogen molecule ion. Proc. Phys. Soc. Sect. A. 62, 519–528 (1949).

The method of the paper reviewed above is applied to the electronic wave function of the hydrogen molecule ion (one electron in the field of two fixed centers of force). A linear combination of atomic orbital wave functions is used as the initial function for the iterative process. The effect of the iterative correction is to broaden the momentum distribution, so that agreement with experiment is improved. The surmise that this will be the case also for other molecules is partially confirmed by a preliminary calculation for the neutral hydrogen molecule. *W. H. Furry.*

Mishra, Brahmananda. Wave functions for excited states of mercury and potassium. Proc. Cambridge Philos. Soc. 48, 511–515 (1952).

The radial wave function $P(nl|r)$ for an electron state (n, l) of the excited states of Hg satisfies the differential equation

$$\left\{\frac{d^2}{dr^2}+\epsilon(nl)+\frac{2Z_p(r)}{r}-\frac{l(l+1)}{r^2}\right\}P(nl|r)=0$$

where ϵ is the energy parameter and $Z_p(r)$ is the total effective nuclear charge. The wave function is normalized so that $\int_0^\infty[P(nl|r)]^2dr=1$. The author computes $Z_p(r)$ from the known values for the neutral atom. He then integrates the differential equation numerically. He uses variables ρ, S defined by $\rho=\log_e(1000r)$, $S(nl|\rho)=r^{\frac{1}{2}}P(nl|r)$ and table 1 gives 4P values of S for the states $6s$, $6p$, $6d$, $7s$,

$7p$, $7d$, and also $2Z_p(r)$, for $\rho=0(1/3)2(1/6)4(1/12)11.25$. Table 3 gives the normalized $P(nl|r)$ for the state $4p$ of K. Here the differential equation contains an additional term $V_p(r)=-\frac{1}{2}\alpha(r^2+r_0^2)^{-2}$ to allow for polarization of the core. The table gives 4D values of $P(4p|r)$ for

$$r=0(.02).3(.5).6(.1)1.2(.2)3.2(.4)16(1)30.$$

Tables 2 and 4 give numerical values of certain integrals.
A. Erdélyi (Pasadena, Calif.).

Wenzl, F. Iterationsverfahren zur Berechnung komplexer Nullstellen von Gleichungen. Z. Angew. Math. Mech. **32**, 85–87 (1952).

Collatz [same Z. **30**, 97–101 (1950); these Rev. **11**, 692] gave a method for the computation of the roots of an algebraic equation with real or complex coefficients. In this paper is given an extension of the method of Collatz to equations not necessarily algebraic. A generalization is also given for systems of equations in more than one variable. The methods are illustrated by examples. *E. Frank.*

Turán, Pál. Approximate solutions of higher algebraic equations. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 279–287 (1951). (Hungarian)

The author discusses Graeffe's method for the approximate solution of algebraic equations. The author's principal result (which he states without proof) is the following: Let $f_0(z)$ be a polynomial of degree n with roots $|z_1|\leqq|z_2|\leqq\cdots\leqq|z_n|$. Define $f_{l+1}(z)=(-1)^n f_l(z^{\frac{1}{2}})f_l(-z^{\frac{1}{2}})$. The roots of $f_\nu(z)$ are $z_1^{2^\nu}, \cdots, z_n^{2^\nu}$. Denote by S_i the sums of the ith powers of the roots of $f_\nu(z)$, i.e. $S_i=\sum_1^n z_k^{i2^\nu}$. Then

$$(1)\qquad \left(\frac{1}{n}\right)^{2^{-\nu}}\leqq|z_n|/(\max_{1\leqq j\leqq 2n}|S_j|^{1/j})^{2^{-\nu}}\leqq 2^{2^{-\nu}}.$$

The author finally discusses various advantages of (1) for practical numerical computations over some previous results of Ostrowski who also obtained explicit inequalities for the goodness of the approximation of Graeffe's method. [Cf. Publ. Math. Debrecen **2**, 26–42 (1951); these Rev. **13**, 77.]
P. Erdös (Los Angeles, Calif.).

Filin, A. P. On the determination of the coefficients in interpolating polynomials. Akad. Nauk SSSR. Inženernyĭ Sbornik **10**, 199–212 (1951). (Russian)

The author derives expressions for the coefficients a_i in the polynomial $y=a_0+a_1x+\cdots+a_mx^m$, where $y=y_i$ when $x=x_i$. The coefficient a_i is expressed in the form

$$a_i=\sum_{j=1}^{m}\Delta_{ij}^{(m)}(y_j-y_0)$$

and tables are provided for $\Delta_{ij}^{(m)}$ for $m=2, 3, 4, 5, 6$, and 7. Several theorems concerning these quantities are given without proof, and a number of auxilliary tables are supplied to facilitate computation. *W. E. Milne* (Corvallis, Ore.).

Rushton, S. On least squares fitting by orthonormal polynomials using the Choleski method. J. Roy. Statist. Soc. Ser. B. **13**, 92–99 (1951).

Let $X_0, \cdots, X_k, Y$ be fixed n-rowed column vectors, and let $\Lambda=(\lambda_j\delta_{ij})$ be an $n\times n$ diagonal matrix of unknowns λ_j. Let X be the n-rowed, $(k+1)$-columned matrix $[X_0X_1\cdots X_k]$. Under suitable hypotheses the best representation $Y=\sum_j\lambda_jX_j=X\Lambda$ is determined by the λ_j for which $|Y-X\Lambda|^2=$ minimum; to find such λ_j one must solve the normal equations $X'X\Lambda=X'Y$. The author recommends solving the normal equations by the Cholesky square-root method [see, for example, Dwyer, Linear computations, Wiley, New York, 1951, p. 113; these Rev. **13**, 283], particularly when X_j is the vector of values of x^j for a given set of abscissas $x_1, \cdots, x_n$—i.e., when we are fitting observations comprising Y by a polynomial of degree k. Let r_j be the sum of squared residuals after fitting a polynomial of jth degree. The author explains the method and shows how the orthonormal polynomials over $x_1, \cdots, x_n$ and the r_j $(j=0, 1, \cdots, k)$ are obtained in the course of the Cholesky solution. There is a numerical example with $n=16$, $k=4$. The author does not count operations, but the reviewer would expect a great saving over a Gram-Schmidt orthogonalization of the vectors X_j. *G. E. Forsythe* (Los Angeles, Calif.).

Fox, L., and Hayes, J. G. More practical methods for the inversion of matrices. J. Roy. Statist. Soc. Ser. B. **13**, 83–91 (1951).

Here, presented for the practical computer, are two arrangements of elimination methods for obtaining A^{-1} after the usual triangular resolution $A=LU$ has already been effected. Both methods feature row-by-row or column-by-column multiplication; one is designed for symmetric matrices. There are four numerical examples of order six.
G. E. Forsythe (Los Angeles, Calif.).

Hayes, J. G., and Vickers, T. The fitting of polynomials to unequally-spaced data. Philos. Mag. (7) **42**, 1387–1400 (1951).

A technique is given for carrying out rapidly the computations involved in fitting a polynomial by least squares to unequally spaced data. The theory behind these methods has been developed by L. Fox, J. G. Hayes, and P. G. Guest in three papers [Fox, J. Roy. Statist. Soc. Ser. B. **12**, 120–136 (1950); Guest, Philos. Mag. (7) **41**, 124–137 (1950); these Rev. **12**, 538; **11**, 692; and the paper reviewed above]. Numerical examples are worked out in detail.
B. Epstein (Detroit, Mich.).

Stearn, J. L. Iterative solutions of normal equations. Bull. Géodésique 1951, 331–339 (1951).

The author describes the cyclic single-step ("Seidel") iterative process for solving a system $Ax=h$, where A is a positive definite matrix. Two numerical illustrations of order 12 are solved; in one of them the Aitken-Steffenson δ^2 process is used to increase the accuracy. For the normal equations of surveying the author has found the Southwell relaxation procedure to require more steps than the above cyclic process. Moreover, he has found the Doolittle (elimination) method to be frequently the most economical and practical of all. With I.B.M. equipment the Doolittle method has been used by the U. S. Coast and Geodetic Survey for systems of orders up to 2300.
G. E. Forsythe (Los Angeles, Calif.).

Fehlberg, E. Bemerkungen zur Konvergenz des Iterationsverfahrens bei linearen Gleichungssystemen. Z. Angew. Math. Mech. **31**, 387–389 (1951).

Consider the Jacobi iterative process

$$\mathfrak{x}_{m+1}=\mathfrak{A}\mathfrak{x}_m+\mathfrak{c}\quad(m=1, 2, \cdots)$$

for finding the solution $\mathfrak{x}_0$ of the finite linear system $\mathfrak{x}=\mathfrak{A}\mathfrak{x}+\mathfrak{c}$. If $d\mathfrak{x}_m=\mathfrak{x}_m-\mathfrak{x}_0$, then $d\mathfrak{x}_{m+1}=\mathfrak{A}\cdot d\mathfrak{x}_m$. 1) The geometric character of the convergence of $\mathfrak{x}_m$ to $\mathfrak{x}_0$ is discussed for various distributions of the eigenvalues λ_i of $\mathfrak{A}$ (assume all $|\lambda_i|<1$). 2) If p is the least integer such that, for all $\mathfrak{x}_1$, $|d\mathfrak{x}_{p+1}|<|d\mathfrak{x}_1|$,

the convergence is defined to be of the "pth kind [Art]." Necessary and sufficient for convergence of the pth kind is that $\mathfrak{E}-(\mathfrak{A}^p)'\cdot\mathfrak{A}^p$ be positive definite, where $\mathfrak{E}$ is the unit matrix. The Jacobi process is always of the pth kind for one definite p; all integers $p=1, 2, \cdots$ are eligible for a certain $\mathfrak{A}=\mathfrak{A}(p)$. 3) The discussion under 2) carries over to the iterative solution of nonlinear systems with non-vanishing Jacobian, when one is sufficiently near a solution.

G. E. Forsythe (Los Angeles, Calif.).

Forsythe, George E., and Motzkin, Theodore S. An extension of Gauss' transformation for improving the condition of systems of linear equations. Math. Tables and Other Aids to Computation 6, 9–17 (1952).

Gauss [Werke, Bd. 9, Teubner, Leipzig, 1903, pp. 250–253, 278–281; for an annotated translation by G. E. Forsythe, see same journal 5, 255–258 (1951); see also R. Dedekind, Gesammelte mathematische Werke, Bd. 2, Vieweg, Braunschweig, 1931, pp. 293–306] suggested the following transformation to improve the convergence of the relaxation method for the solution of a system of linear equations (1) $\sum a_{ij}x_j=b_i$ $(i=1, 2, \cdots, n)$ of which the matrix (a_{ij}) is symmetric and positive definite. Replace x_i by $y_i-y_{n+1}(i=1, \cdots, n)$ and adjoin to the system an $(n+1)$th equation by adding the first n. The new system is redundant but this does not matter since the x_i's are determined by the differences of the y_i's. Interest in this transformation was roused by its description by Zurmühl [Matrizen . . . , Springer, Berlin-Göttingen-Heidelberg, 1950, p. 280; these Rev. 12, 73]. The authors study a generalization of the above transformation: they consider $x_i=y_i+s_i\,y_{n+1}$ where the s_i are parameters. They also consider the case when (a_{ij}) is singular.

Let λ_i be the characteristic root of (a_{ij}). Let $\lambda=\max|\lambda_i|$ and let $\mu=\min|\lambda_i|$ where we disregard vanishing λ_i. It is suggested that the size of the ratio λ/μ, called the P-condition number, is decisive in the speed of convergence of iterative solutions of (1). [Cf. von Neumann and Goldstine, Bull. Amer. Math. Soc. 53, 1021–1099 (1947); these Rev. 9, 471; Turing, Quart. J. Mech. Appl. Math. 1, 287–308 (1948); these Rev. 10, 405; J. Todd, ibid. 2, 469–472 (1949); Proc. Cambridge Philos. Soc. 46, 116–118 (1950); these Rev. 11, 619, 403.] It is shown that the characteristic roots of the transformed matrix separate those of the original and that although values of the s_i can be chosen to reduce the P-condition number it is not true that the original Gauss transformation will always effect this. The case of a well-known 4×4 ill-conditioned matrix is discussed in numerical detail.

J. Todd (Washington, D. C.).

Bodewig, E. Bericht über die Methoden zur numerischen Lösung von algebraischen Eigenwertproblemen. I. Atti Sem. Mat. Fis. Univ. Modena 4, 133–193 (1950).

This is the first half (iterative methods) of a report intended to include all known numerical methods of finding the eigenvectors and eigenvalues of finite matrices A. The complete report would compare with the author's comprehensive survey of methods of solving systems of linear equations [Nederl. Akad. Wetensch., Proc. 50, 930–941, 1104–1116, 1285–1295 (1947); 51, 53–64, 211–219 (1948) = Indagationes Math. 9, 441–452, 518–530, 611–621 (1947); 10, 24–35, 82–90 (1948); these Rev. 9, 250, 382, 621]. As in the earlier report, many methods are described, with frequent references to a bibliography (but see end of this review). The number of required operations (multiplications and recordings) is counted, and is generally adopted as an (inverse) measure of the utility of the method.

In the introduction there is a historical survey of direct methods by Leverrier (1840), Krylov (1931), Danilewski (1937), Frazer-Duncan-Collar (1938), Hessenberg (1940), Samuelson (1942), and R. Weber (1949). Although the present paper deals in detail only with iterative methods, the author's judgment is clear: direct methods are better because they require fewer operations.

There is a section on algebraic fundamentals, in which nonlinear elementary divisors are dealt with adequately. After Gerschgorin's theorem, the report on iterative methods begins with matrix powering, iteration on a vector, and convergence acceleration with inner products and the δ^2 process. For getting the remaining eigenvalues one may use: (1) minors of the recurrent determinant of moments, (2) various ways of deflating the matrix A, (3) orthogonalization of the vector iterates, or (4) polynomials in A. The report continues with methods involving the solution of systems of linear equations approximating the non-linear system $Ax-tx=0$; there is a diatribe on the contributions of the Southwell school. We next read of Jacobi's method of rotating symmetric matrices into diagonal form, and Bodewig's generalization thereof. The paper concludes with some perturbation methods for diagonalization.

The paper contains a wealth of information based on the author's wide reading and analysis; it is especially valuable for the beginner in the field. But the reader must beware of unreliable statements. For example, (p. 164) smaller exponential terms do *not* interfere with the elimination by the δ^2 process of the dominant exponential bq^n. And again: the process of p. 174 does *not* ordinarily converge quadratically unless A is symmetric. There are many minor misprints. The reviewer noted no omissions of methods known at the time. More recently, however, gradient methods for minimizing the Rayleigh quotient $(Ax, x)/(x, x)$ (A symmetric) have been much exploited; see, for example, Hestenes and Karush [J. Research Nat. Bur. Standards 47, 45–61 (1951); these Rev. 13, 283].

Operation counts, while valuable, seem to be overemphasized. For modern machinery simplicity of coding, economy of storage, and stability are usually more important than the number of operations. Even for desk machines (which is all the author claims to deal with), the accuracy of the answer achieved is very relevant. It surely can happen, for instance, that by a theoretically efficient direct method one gets the characteristic polynomial with intolerable round-off errors, whereas a theoretically less efficient iterative process succeeds in getting all that is desired.

The paper appears to be a copy of a mimeographed version privately circulated by the author. If so, in the transcription the Table of Contents of the entire paper and the valuable bibliography were unfortunately omitted, and some internal page references were left unchanged.

G. E. Forsythe (Los Angeles, Calif.).

Lopšic, A. M. A numerical method for finding the characteristic values and characteristic planes of a linear operator. Trudy Sem. Vektor. Tenzor. Analizu 7, 233–259 (1949). (Russian)

The paper is concerned with finding all characteristic values of an arbitrary real matrix A of order n and the corresponding invariant manifolds. The method is that wherein starting with an arbitrary vector a the first m linearly independent vectors $b_k=A^ka$ are formed: $\sum_{k=0}^{m}\alpha_kb_{m-k}=0$.

It is shown that $m<n$ is to be expected when n is sufficiently large and computations are performed with a fixed degree of accuracy. Development of suitable procedures to be employed when $m<n$, and appropriate geometric formulation of the situations that arise, constitute the principal part of the paper. The better known case $m=n$ is also fully treated. Two methods are given for finding m and the α_k for any given sequence of vectors b_k. They are compared by enumeration of operations and by consideration of phenomena which occur in applying them to the present problem. The difference quotients of successive orders for the polynomial $\varphi(\lambda)=\sum_{k=0}^{m}\alpha_k\lambda^{m-k}$ play a prominent rôle throughout the paper which is very closely related to but not dependent upon the work of K. A. Semendiaev [Akad. Nauk SSSR. Prikl. Mat. Meh. 7, 193–222 (1943); these Rev. 6, 51].

R. Church (Monterey, Calif.).

Fortet, R. On the estimation of an eigenvalue by an additive functional of a stochastic process, with special reference to the Kac-Donsker method. J. Research Nat. Bur. Standards 48, 68–75 (1952).

A sampling ("Monte Carlo") method is described for the computation of the Fredholm determinant, and hence of the eigenvalues, of certain Fredholm integral equations with positive definite kernel $\Gamma(t,\tau)$. The method is based on a theorem by Kac and Siegert [J. Appl. Phys. 18, 383–397 (1947); these Rev. 8, 522] which connects the Fredholm determinant of the integral equation with a Laplacian (i.e. Gaussian) process $X(t)$ whose covariance is $\Gamma(t,\tau)$. Such a process $X(t)$ is, in general, not easy to find. In the special case that $\Gamma(t,\tau)$ depends on $t-\tau$ only an appropriate $X(t)$ can be constructed by means of a Poisson process. The second part of the paper contains a discussion of the various errors inherent in the method of Donsker and Kac [cf. same J. 44, 551–557 (1950); these Rev. 13, 590] for the determination of the lowest eigenvalue of Schrödinger's equation. [Reviewer's note: In an unpublished paper by M. Cohen and M. Kac it is shown that some of the bounds for the errors given by Fortet are wasteful by as much as a factor 5000.]

W. Wasow (Los Angeles, Calif.).

Kruse, U. E., and Ramsey, N. F. The integral

$$\int_0^\infty y^2 \exp(-y^2+ix/y)dy.$$

J. Math. Physics 30, 40–43 (1951).

In various problems of theoretical physics there arise definite integral functions which can be reduced to the function

$$\varphi_1(x)=\int_0^\infty \exp(-y-x\sqrt{y})ydy,$$

studied by C. T. Zahn [Physical Rev. 52, 67–71 (1937)] and O. Laporte [ibid. 52, 72–74 (1937)]. For values of $x<3$ Zahn has developed a convergent series. For larger values of x it is more convenient to use the asymptotic expansion due to Laporte

$$\varphi_1(x)=2(\pi/3)^{\frac{1}{2}}(x/2)\exp-[(3/2)(x/2)^{2/3}]\sum_{p=0} K_p(2/x)^{(2/3)p},$$

where $K_0=1$,

$$36pK_p=(-12p^2+12p+35)K_{p-1} \\ -(1/6)(5+2p)(4-2p)(3-2p)K_{p-2}.$$

A table is calculated for the related functions

$$I(x)=\int_0^\infty \exp(-y^2)y^2\cos(x/y)dy=\tfrac{1}{2}\operatorname{Re}\{\varphi_1(ix)\},$$

$$K(x)=\int_0^\infty \exp(-y^2)y^2\sin(x/y)dy=z-\tfrac{1}{2}\operatorname{Im}\{\varphi_1(ix)\}$$

by means of these expansions for $x=0(.1)1(.2)8(.5)20$, in five decimals for $x=0$ to 2, 7 to 20, in four decimals for $x=2$ to 7.

S. C. van Veen (Delft).

Štykan, A. B. Graphical solution of differential equations with a deviated argument. Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 184–191 (1952). (Russian)

The author gives approximate graphical solutions for equations of the type

$$y'(x)=F[x, y(x), y(x\pm\alpha(x)), y(x\pm\beta(y))]$$

and systems of such equations. The theory of such equations has been studied by Myškis [see, e.g., Uspehi Matem. Nauk 4, no. 5(33), 99–141 (1949) = Amer. Math. Soc. Translation no. 55 (1951); these Rev. 11, 365; 13, 752].

J. M. Danskin (Santa Monica, Calif.).

Azbelev, N. V. On an approximate solution of ordinary differential equations of the nth order based upon S. A. Čaplygin's method. Doklady Akad. Nauk SSSR (N.S.) 83, 517–519 (1952). (Russian)

The author gives theorems providing upper and lower bounds for the solution of an nth order ordinary differential equation found by the method of Čaplygin.

W. E. Milne (Corvallis, Ore.).

Maruašvili, T. I. On convergence of sequences of approximate values of critical forces constructed by a difference method. Soobščeniya Akad. Nauk Gruzin. SSR. 9, 83–89 (1948). (Russian)

The author considers a beam of variable stiffness pinned at both ends and subject to a force of compression P. The system to be solved is (*) $EI(x)y''=-Py$ with $y(0)=y(1)=0$. He replaces (*) by a difference equation set up for n values of y at n equally spaced values of x with interval h. This leads to a set of homogeneous linear equations for the n values y_i, which can have solutions only for particular values of P, $P_k^{(n)}$. The aim of the paper is to show that as n becomes infinite $P^{(n)}$ converges to a critical force $\bar{P}$, and the corresponding $y_i^{(n)}$ to $y(x)$, when $y(x)$ is a non-zero solution of (*) corresponding to $\bar{P}$. *W. E. Milne* (Corvallis, Ore.).

Maruašvili, T. I. An estimate of the error of the critical forces of a compressed bar computed by a difference method. Soobščeniya Akad. Nauk Gruzin. SSR. 9, 145–152 (1948). (Russian)

In this paper, following the ideas of Collatz, the author uses the Rayleigh quotient to set up inequalities giving an estimate of the error in the approximation to the lowest critical force, which he shows to be $O(h^4)$. *W. E. Milne.*

Hildebrand, F. B. On the convergence of numerical solutions of the heat-flow equation. J. Math. Physics 31, 35–41 (1952).

A proof is given for the convergence of the solution of the simplest type of finite difference representation of the one-dimensional heat flow equation to the solution of the governing differential equation. The basic proof is carried out for a

finite slab, for any mesh ratio $0<\Delta t/(\Delta x)^2<\frac{1}{2}$, and for an initial heat distribution which is sectionally continuous (and of bounded variation). An essential feature of the proof is that all members of the sequence of finite difference solutions used in the limiting process satisfy the initial conditions at the mesh points. At the conclusion of his paper the author gives an illustration of an "unstable" finite difference equation which converges. It should be noted, however, that in doing so he implies a somewhat different definition of stability than that used in the paper by O'Brien, Hyman and Kaplan [same J. 29, 223–251 (1951); these Rev. 12, 751]. The reviewer is in agreement with the author's statement that instability does not necessarily imply absence of strict convergence. This is true irrespective of which of the two definitions of stability is used. *H. Polachek.*

Hyman, Morton A. Non-iterative numerical solution of boundary-value problems. Appl. Sci. Research B. 2, 325–351 (1952).

It is shown that the numerical solution by finite difference methods of boundary value problems for elliptic differential equations can frequently be simplified by combining the use of an explicit solution of the difference problem as obtained by finite Fourier analysis with the step-by-step procedures that are normally used for initial value problems only. Since the numerical evaluation of the explicit solution is often cumbersome, it is proposed that it be carried out for certain pairs of "starting lines" only. At the remaining inner net points the solution is then obtained by the step-by-step procedure. The author indicates how to choose these starting lines so as to minimize the unavoidable numerical errors. A number of methods are suggested by means of which the explicit solution at the starting lines can be calculated more easily. It is shown that such explicit solutions in terms of trigonometric and exponential functions can be obtained for boundary problems in rather general domains, although they are, of course, best adapted to problems in rectangles, with which they are usually associated. *W. Wasow.*

Novožilov, V. V. On an approximate method of solution of boundary problems for ordinary differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 305–318 (1952). (Russian)

Given the differential equation $L(y)+f(x, y)=F(x)$, with two-point boundary conditions at a and b, L being a linear differential operator, the author seeks a suitable y_1 for starting the iteration $L(y_{i+1})=F(x)-f(x, y_i)$, assumed convergent. He takes $y_1=a_1\varphi_1(x)+\cdots+a_m\varphi_m(x)$, with convenient φ's, determining the a's to minimize the integral square of $y_2^{(k)}-y_1^{(k)}$, for whatever k is deemed appropriate to the problem. Admitting the lack of rigorous justification, the author comments that "if one speaks not of the exception, but of the rule, the considerations adduced above appear sufficiently convincing," and works out three numerical examples. *A. S. Householder* (Oak Ridge, Tenn.).

Gombás, P., und Gáspár, R. Zur Lösung der Thomas-Fermi-Diracschen Gleichung. Acta Phys. Acad. Sci. Hungaricae 1, 66–74 (1951). (German. Russian summary)

The basic differential equation of the Fermi-Thomas-Dirac statistical model of the atom was solved numerically by K. Umeda [J. Fac. Sci. Hokkaido Univ. Ser. II. 3, 171–244 (1942), 245 (1949); these Rev. 13, 872] for neutral atoms from $Z=1$ to $Z=92$. The boundary conditions proposed by Brillouin and applied by Umeda do not correspond to the physical requirements of the problem as well as the somewhat different boundary conditions suggested by Jensen. By a perturbation method, the authors obtain from Umeda's results the results for Jensen's boundary conditions, in the five cases $Z=10$, 18, 36, 54, 86. This method is also modified to obtain solutions for singly and doubly ionized atoms and for compressed atoms. *W. H. Furry.*

Metropolis, N., and Reitz, J. R. Solutions of the Fermi-Thomas-Dirac equation. J. Chem. Phys. 19, 555–573 (1951).

Solutions obtained from the ENIAC electronic computer are given for 24 different values of Z, distributed roughly uniformly from $Z=6$ to $Z=92$. For each of the 24 cases the function ψ is tabulated for six different values of the initial slope. Each set of values of ψ is continued until either ψ' satisfies Jensen's boundary condition for a neutral atom, or else ψ becomes negative. The authors do not apply Jensen's boundary condition on ψ; instead, they regard the solutions for a given Z as corresponding to different sizes of the atom, or else to different degrees of ionization. The condition $\psi(x_0)=0$ used in the latter type of case is indeed practically indistinguishable from Jensen's condition, especially for large Z. Atomic radii and degrees of ionization are stated numerically for the various cases. The degrees of ionization have nonintegral values, and are of the order of 10 for the larger values of Z. *W. H. Furry* (Cambridge, Mass.).

✶Souriau, J. M. Une méthode générale de linéarisation des problèmes physiques. Actes du Colloque International de Mécanique, Poitiers, 1950. Tome IV. Études sur la mécanique des solides, études sur la mécanique générale, pp. 251–268. Publ. Sci. Tech. Ministère de l'Air, no. 261, Paris, 1952.

Luzzati, V. Sur la convergence et l'erreur dans les structures non-centrosymétriques. Acta Cryst. 4, 367–369 (1951).

L'auteur analyse les résultats obtenus par Cruickshank [Acta Cryst. 3, 10–13 (1950); ces Rev. 13, 591] dans la détermination des structures cristallines. En employant la méthode plus directe de l'application réiterée des transformations de Fourier, il trouve dans le cas des structures non-centrosymétriques, que à partir d'une distance suffisamment petite des positions correctes des atomes, le procédé habituel de l'application réiterée des transformations de Fourier déplace chaque pic de la transformée indépendamment vers sa position correcte. Pour rendre la méthode plus rapidement convergente, à chaque approximation il faudrait déplacer les atomes vers les nouvelles positions, à une distance deux fois plus grande que celle indiquée par la transformée de Fourier. Ceci confirme la conclusion de Cruickshank, à savoir que toutes choses égale, d'ailleurs, les positions atomiques déterminées par la méthode des transformations de Fourier dans les structures non-centrosymétriques sont affectées par des erreurs deux fois plus grandes que celles obtenues dans les structures avec centre de symétrie. *S. C. van Veen* (Delft).

Sokoloff, Boris Alexandre. Principe et réalisation d'une machine mathématique dite "Opérateur Mathématique Electronique." Ann. Télécommun. 5, 143–159 (1950).

This paper describes a device for solving linear differential equations with constant coefficients. Computing circuits consisting of condensers and potentiometers are used as feedback circuits for electronic amplifiers. The stability of

such an arrangement is studied. The device can also be used for solving simultaneous algebraic linear equations, by steady state considerations. As a differential analyzer two outputs are described. One involves a cathode ray tube, the other seems to be equivalent of a servo driven output table. The last is also used as a function input table. In an application to Van der Pol's equation, a servo multiplier is mentioned. *F. J. Murray* (New York, N. Y.).

Mitrovic, Dusan. Sur un principe nouveau de construction des machines électriques servant pour la recherche des racines des équations algébriques. C. R. Acad. Sci. Paris **234**, 2519–2521 (1952).

de Beauclair, W. Der Sonderschieber für Häufigkeitsrechnung. Z. Angew. Math. Mech. **32**, 112–120 (1952). (German. English, French and Russian summaries)

The author describes a newly-constructed special slide rule for statistical computations.

From the author's summary.

Fil'čakov, P. F. Modelling of filtration problems on electrically conducting paper. Doklady Akad. Nauk SSSR. (N.S.) **84**, 237–240 (1952). (Russian)

✱Barrois, W. Applications du transfert de contact à l'étude des problèmes linéaires: systèmes hyperstatiques et vibrations. Actes du Colloque International de Mécanique, Poitiers, 1950. Tome IV. Études sur la mécanique des solides, études sur la mécanique générale, pp. 269–314. Publ. Sci. Tech. Ministère de l'Air, no. 261, Paris, 1952.

✱Couffignal, L. Sur l'emploi des grosses machines à calculer. Actes du Colloque International de Mécanique, Poitiers, 1950. Tome IV. Études sur la mécanique des solides, études sur la mécanique générale, pp. 205–209. Publ. Sci. Tech. Ministère de l'Air, no. 261, Paris, 1952.

Everett, R. R. The Whirlwind I computer. Elec. Engrg. **71**, 681–686 (1952).

Govorkov, V. A. The computation of electrical and magnetic fields by the method of a potential grid. Električestvo **1949**, no. 3, 52–56 (1949). (Russian)

Govorkov, V. A. The computation of electrical and magnetic fields in polar coordinates by the method of a potential grid. Električestvo **1951**, no. 7, 51–58 (1951). (Russian)

Exposition of simple methods of successive adjustments of numerical values at meshpoints. *R. Church.*

✱Malavard, L. Aperçu sur la méthode d'analogie rhéoélectrique. Actes du Colloque International de Mécanique, Poitiers, 1950. Tome IV. Études sur la mécanique des solides, études sur la mécanique générale, pp. 179–203. Publ. Sci. Tech. Ministère de l'Air, no. 261, Paris, 1952.

Nikolaev, P. V. On the degree of the basic curves of a nomogram. Doklady Akad. Nauk SSSR (N.S.) **76**, 353–354 (1951). (Russian)

Suppose that $F(t_1, t_2, t_3)=0$ is an algebraic M-equation, that is, there exists an anamorphizing factor of the form

$$\psi_1(t_2, t_3)\psi_2(t_3, t_1)\psi_3(t_1, t_2)$$

whose product with $F(t_1, t_2, t_3)$ is a polynomial expressible as a Massau determinant. It is shown that in this case the dimension of $F(t_1, t_2, t_3)=0$ in each variable t_i [Nikolaev, same Doklady **67**, 421–423 (1949); these Rev. **11**, 406] is one less than the degree of the common basis of the three variables. If the anamorphizing factor is of the form $\psi_3(t_1, t_2)$, then the dimension in t_1 (and t_2) is equal to the degree of the common basis of the variables t_1 and t_2. *R. Church.*

Nikolaev, P. V. On nomographing algebraic equations. Doklady Akad. Nauk SSSR (N.S.) **76**, 489–491 (1951). (Russian)

Making use of his earlier work [especially Mat. Sbornik N.S. 17(59), 253–266 (1945); these Rev. **7**, 489], the author first gives necessary conditions for the existence of an anamorphizing factor containing one variable for the algebraic M-equation $F(t_1, t_2, t_3)=0$ [see the preceding review]. A method for determination of the anamorphizing factor (involving any of the variables) and the Massau determinant is outlined for such equations. It is shown that an algebraic M-equation with real coefficients has a real nomogram.

R. Church (Monterey, Calif.).

RELATIVITY

Majorana, Quirino. Nuove considerazioni cinematiche sulla relatività speciale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 245–251 (1952).

Hlavatý, Václav. The Einstein connection of the unified theory of relativity. Proc. Nat. Acad. Sci. U. S. A. **38**, 415–419 (1952).

In this second very short note a complete solution $\Gamma^{\nu}_{\lambda\mu}$ of the equations of the unified theory is given and this is applied to the first set of Einstein's conditions. Detailed proofs will be given in another publication. In the conclusion some physical interpretations are mentioned.

J. A. Schouten (Epe).

Bandyopadhyay, G. Particular solutions of Einstein's recent unified theories. Indian J. Phys. **25**, 257–261 (1951).

The "hermitian" generalized field equations of Einstein and Straus [Ann. of Math. (2) **47**, 731–741 (1946); these Rev. **8**, 412] and those later proposed by Einstein [The meaning of relativity, 3rd ed., Princeton Univ. Press, 1950, appendix II] have been solved by assuming that the nonsymmetric tensor g_{ij} is static and invariant under the group of rotations and translations of a plane. It is shown that in the former case fields corresponding to both a plane distribution of charged masses and massless charges exist whereas in the latter case only electric fields corresponding to a plane distribution of massless charges exist.

A. H. Taub (Urbana, Ill.).

Kurşunoğlu, Behram. Einstein's unified field theory. Proc. Phys. Soc. Sect. A. 65, 81–83 (1952).

The author shows that in Einstein's new theory the energy momentum tensor vanishes identically. His result is based on Einstein's field equations, and also on another assumption which the author himself realizes is to some extent arbitrary. A generalized version of Einstein's new theory is sketched, in which the energy momentum tensor exists. The author expects that the equations of motion can be derived from these field equations. *L. Infeld* (Warsaw).

Linés Escardó, Enrique. On relativistically invariant functional products. Rivista Mat. Univ. Parma 2, 215–234 (1951). (Spanish)

The author first uses the invariance of the projective trace of a function under linear homogeneous transformations of the variables to calculate the trace of a function depending on a bilinear form in the variables by reducing the bilinear form to canonical form. Secondly he establishes a rule for the reduction of repeated projective traces with respect to several distinct sets of variables to one single trace with respect to all the variables. Using these two results, he obtains in a simple form the projective indicatrices of the relativistically invariant functional products introduced by Fantappiè [Ann. Mat. Pura Appl. (4) 29, 43–69 (1949); these Rev. 13, 501]. *A. J. McConnell* (Dublin).

Synge, John L. Sur les connections relativistes entre la fréquence, la longueur d'onde, la vitesse de phase et la vitesse de groupe. C. R. Acad. Sci. Paris 234, 1669–1670 (1952).

L. de Broglie [same C. R. 225, 361–363 (1947); these Rev. 9, 258] assuming relativistic invariance, derived formulas relating ν (frequency), μ (reciprocal of wave length), V (phase velocity), and U (group velocity). These formulas contained an unspecified function $U(\mu)$. In this note the author shows that relativistic invariance implies more; in particular, the function $U(\mu)$ must have the form $c\mu/(\mu^2+A)^{\frac{1}{2}}$, and $V(\mu)$ the form $c(\mu^2+A)^{\frac{1}{2}}/\mu$, where A is a constant and c is the velocity of light. *O. Frink.*

Stiegler, Karl Drago. Sur le principe de la constance de la vitesse de la lumière. C. R. Acad. Sci. Paris 234, 1250–1252 (1952).

The Lorentz transformation can be deduced from three axioms none of them stating the constancy of the velocity of light. *L. Infeld* (Warsaw).

Ingraham, Richard L. L'ennuple projectif et l'unification des théories de l'électromagnétisme de Weyl et de Veblen-Hoffmann. Ann. Inst. H. Poincaré 12, 131–158 (1951).

The author defines a covariant differentiation of two component spinors in projective relativity and then derives the field equations obtained from the Lagrangean used by Veblen and Hoffmann [Physical Rev. 36, 810–822 (1930)] modified by the addition of a term similar to that proposed by Weyl [Z. Physik 56, 330–352 (1929)]. These field equations reduce in the case of no gravitation to Maxwell's equations and the Dirac equations in terms of two component spinors for a particle of zero mass. *A. H. Taub.*

Goto, K. Wave equations in de Sitter space. Progress Theoret. Physics 6, 1013–1014 (1951).

The author uses a representation of de Sitter space as a four-dimensional "hypersphere" in a five-dimensional flat space:

$$x_1^2+x_2^2+x_3^2-x_4^2+x_5^2=R^2. \tag{1}$$

The wave equations proposed are

$$\{a^{\mu\nu}m_{\mu\nu}-2RK\}\psi=0 \tag{2}$$

where $m_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu$, K is a constant, and $a^{\mu\nu}$ are matrices corresponding to the infinitesimal generators of the five-dimensional group leaving (1) invariant. Equation (2) is evaluated for various representations in the neighborhood of the point $(0, 0, 0, 0, R)$ of the de Sitter space. *A. H. Taub* (Urbana, Ill.).

MECHANICS

★Dobrovol'skiĭ, V. V. Teoriya mehanizmov. [Theory of mechanisms.] Gosudarstv. Naučno-Tehn. Izdat. Mašinostroit. Lit., Moscow, 1951. 465 pp.

This is a first, but remarkably comprehensive course, with the following distinctive features. Traditional geometry of motion is treated only in the context, and limited to the bare essentials (e.g., no mention of the Euler-Savary theorem could be found). The dynamics (with friction accounted for) and kinetostatics of each group of mechanisms is discussed jointly with its kinematics. The author's classification of mechanisms (widely favored in USSR) is presented in full detail (55 pp.) and used to organize the material. Some recent results (especially the author's, e.g., formulas for transmission efficiency) are included, but no topic is treated on the design specialist's level. Space mechanisms are given little attention in spite of the author's work in this field. The basic idea of the classification is: if, after the kinematic pairs are disconnected, the rigid members have m degrees of freedom, the mechanism is of class $m-1$, and the Gruebler formula is replaced by $mn-\sum_k(m-k)p_k$, $k=1, \cdots, m-1$, if n is the number of links and there are no passive constraints. Two-link mechanisms are assigned to class zero. *A. W. Wundheiler* (Chicago, Ill.).

★Deimel, Richard F. Mechanics of the gyroscope. The dynamics of rotation. Dover Publications, Inc., New York, N. Y., 1950. ix+192 pp. Paperbound, $1.60; clothbound, $3.25.

Reprinted with correction of errors from the first edition [Macmillan, New York, 1929].

Nadile, Antonio. Vibrazioni con ereditarietà dei sistemi olonomi a due gradi di libertà. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 85, 9–21 (1951).

The author studies a dynamical system of two degrees of freedom, subject to resistance which accumulates from a specified time with exponential coefficients. He concludes that the motion is equivalent to the superposition of six exponential motions, or of four exponentials and one damped oscillation, or of two exponentials and two damped oscillations. He considers also the case of a sinusoidal driving force on one of the degrees of freedom. The results generalize those of D. Graffi [Nuovo Cimento (8) 5, 310–317 (1928)] for a single degree of freedom. *C. Truesdell* (Bloomington, Ind.).

de Castro Brzezicki, A. On the small oscillations of a mass point about the lowest point of a surface. Gaceta Mat. 2, 107–114 (1950). (Spanish)

The behavior of small oscillations about a stable point of equilibrium are discussed once more by the classical method of linearizing the differential equations of motion.

F. Bohnenblust (Pasadena, Calif.).

Šul'gin, M. F. On the theory of the Lagrange equations for nonconservative systems. Doklady Akad. Nauk SSSR (N.S.) 83, 373–376 (1952). (Russian)

This paper is a continuation of an earlier one by the author [same Doklady 81, 23–26 (1951); these Rev. 13, 594], and establishes a theorem, analogous to the classical theorem of Poisson, on integrals of Lagrangian equations for non-conservative systems. By the introduction of superfluous variables the given Lagrangian system is replaced by an extended one. Given an integral of each system, the theorem provides a new integral of the original one. If the given integral of the original system is linear with respect to the velocities, the corresponding extended system can be reduced to the Lagrangian form with respect to the superfluous variables. *E. Leimanis* (Vancouver, B. C.).

Šul'gin, M. F. Poisson's theorem for the equations of dynamics with coupling factors. Doklady Akad. Nauk SSSR. (N.S.) 84, 453–456 (1952). (Russian)

The method of integration based on successive application of the classical Poisson theorem is extended to the equations of motion of a holonomic conservative system with coupling factors. *E. Leimanis* (Vancouver, B. C.).

Sokolov, Yu. D. On the motion of a system of three material points on a straight line. Ukrain. Mat. Žurnal 1, no. 3, 3–40 (1949). (Russian)

The author considers the straight-line motion of three material points with masses m_0, m_1, m_2, attracting (or repulsing) each other according to the forces with moduli

$$m_i m_j |f(r_k)| \quad (i, j, k=0, 1, 2; i\neq j\neq k),$$

where r_k is the distance between the masses m_i, m_j and $f(r)$ is an analytic function, positive in the case of repulsion and negative in the case of attraction. The author discusses the integrability by quadratures of the equations of motion, double collision between a definite pair of material points at a definite point, while the third material point approaches a definite distinct point, as well as the triple collision at the common center of gravity, taking place at a finite instant under the assumption

$$\lim_{r\to 0} r^{2\alpha+1}f(r)=\pm 2\alpha$$

($\alpha\neq 0$ an arbitrary real number). Moreover the behavior of motion for a boundless increasing of the moment of inertia of the system, assuming that

$$\lim_{r\to\infty} f(r)/r^{2\alpha'-1}=2\alpha'>2,$$

is considered. The paper terminates with a discussion of motion in which during a finite time interval the mutual distances of the three points increase indefinitely.

E. Leimanis (Vancouver, B. C.).

Sokolov, Yu. D. On a general case of symmetric motion of a system of three material points. Ukrain. Mat. Žurnal 2, no. 3, 7–44 (1950). (Russian)

This paper treats the case of symmetric motion in a space of three material points mutually attracting (or repulsing) according to forces with moduli $m_i m_j |f(r_{ij})|$, $i, j=0, 1, 2$; $i\neq j$. After the discussion of the straight line and homographic motion it is shown that in the general case of a non-homographic motion for an arbitrary $f(r)$ the isosceles triangle $P_2P_0P_1$, formed by the three material points, can only rotate about its altitude or about the axis parallel to its base through the center of inertia of the system. The cases of integrability of the equations of motion in elementary and elliptic functions are given. In the second part of the paper double and triple collisions, unlimited recession in a finite time of the moving points and the analytical representation of the solutions in the neighborhood of $r=0$ and $r=\infty$ under some restrictions on $f(r)$ are discussed.

E. Leimanis (Vancouver, B. C.).

Sokolov, Yu. D. On rectilinear motion with a common collision of a system of three material points mutually attracting according to an exponential law. Ukrain. Mat. Žurnal 2, no. 4, 18–24 (1950). (Russian)

The modulus of the law of attraction specified in the title is $g^2 m_i m_j e^{a/r_k}$ $(i, j, k=0, 1, 2; i\neq j\neq k)$, g^2 and a being positive constants. The case of a triple collision, taking place at a finite instant is investigated and the corresponding analytical representation is discussed. *E. Leimanis.*

Sokolov, Yu. D. On the motion on a straight line of a system of three material points, each acting with forces proportional to the logarithms of their mutual distances. Ukrain. Mat. Žurnal 2, no. 4, 25–36 (1950). (Russian)

The rectilinear motion of the three points, mutually attracting or repulsing, is assumed to take place under the action of the forces with moduli $g^2 m_i m_j |\ln r_k/a|$ $(i, j, k=0, 1, 2; i\neq j\neq k)$. Double collision between a definite pair of the points and the triple collision taking place at a finite instant are considered and the corresponding analytical representations are discussed. *E. Leimanis* (Vancouver, B. C.).

Ryabov, G. A. On the stability of particular solutions of the three body problem. Akad. Nauk SSSR. Astr. Žurnal 29, 341–349 (1952). (Russian)

The author investigates the problem of orbital stability of the straight-line solutions of the general problem of three bodies. By application of the so-called second method of Lyapunov it is shown that strong orbital instability takes place provided the eccentricities of the orbits, described by the moving points, do not exceed some finite limit. This limit is estimated approximately. *E. Leimanis.*

Grémillard, Jean. Sur les solutions périodiques de la troisième sorte dans le problème des trois corps. C. R. Acad. Sci. Paris 234, 2339–2341 (1952).

Charlier contested the existence of certain periodic solutions of the third kind, obtained by Poincaré and corresponding to initial eccentricities zero. Starting from Poincaré's equations the author shows that such solutions of the problem of three bodies in fact exist and gives the conditions for their existence. *E. Leimanis* (Vancouver, B. C.).

Pedersen, Peder. Stabilitätsuntersuchungen im restringierten Vierkörperproblem. Danske Vid. Selsk. Mat.-Fys. Medd. 26, no. 16, 37 pp. (1952).

This paper is a continuation of a previous one [same journal 21, no. 6 (1944); these Rev. 7, 493] in which the author determined the location of equilibrium points for the planar restricted 4-body problem, in which an infinitesimal mass is attracted by three finite masses moving according

to an equilateral triangle solution of the three-body problem. The stability of the equilibrium points (8, 9, or 10 in number) is investigated and the plane is subdivided into regions of different types of stability or instability; the boundaries of the regions are computed with considerable accuracy.
W. Kaplan (Ann Arbor, Mich.).

Carrière, P. Perturbations balistiques d'un projectile autopropulsé à poudre, pendant la phase d'autopropulsion. Mémorial de l'Artillerie Française 25, 253–360 (1951).

This long paper handles a special but increasingly important part of theoretical exterior ballistics, as may be inferred from its title. It deals with both fin-stabilized and spin-stabilized missiles during propulsion in air. Particular discussion is devoted to effects of (a) an initial impulse couple, (b) lateral wind, and (c) eccentricity of propulsion with respect to the center of mass. A qualitative study is made of the behavior of the projectile at transsonic velocities. A number of detailed and informative graphs, and various auxiliary mathematical tables (to four significant figures) are furnished. As the author points out, certain reasonable simplifying assumptions are adopted. The aim is to provide the ballistician with means for predicting the nature and magnitude of ballistic perturbations, and thus to equip him to take measures a priori to reduce these to a minimum. In illustrative problems specific weapons are named and corresponding numerical data provide initial conditions. Unfortunately, the reader finds little evidence that the behavior predicted by so many formulas and with such exactness is substantiated by observation. Possibly security regulations forbid publication of results, to obtain which would at best require very elaborate equipment.
A. A. Bennett (Providence, R. I.).

Hydrodynamics, Aerodynamics, Acoustics

Jarre, Gianni. Sul moto relativo nei mezzi continui. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 85, 183–191 (1951).

Vectorial proofs of simple known results on relative motion. *C. Truesdell* (Bloomington, Ind.).

Manera, Giancarlo. Il principio della conservazione dell'energia, secondo Lagrange, nei liquidi in riposo. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 85, 348–354 (1951).

The author's equation (5) is $\rho d\phi = dp$, which is equivalent to the equation of hydrostatics. The remainder is valid only subject to the unstated assumption that compressibility ϵ is the same at all densities. The result is an expression for $d\phi$ in terms of ϵ and dp. Thus the title is rather misleading.
C. Truesdell (Bloomington, Ind.).

Kampé de Fériet, Joseph, et Kotik, Jack. Sur les ondes de pesanteur à deux dimensions d'énergie finie. C. R. Acad. Sci. Paris 235, 230–232 (1952).

Let $F(x)$, defined for $-\infty < x < \infty$, satisfy: (a) $F(-x) = F(x)$; (b) $F(x) \epsilon L^2$; (c) $F(x)$ is absolutely continuous; (d) $F'(x) \epsilon L^2$. Let

$$f(\lambda) = \text{l.i.m.}_{A\to\infty} (-2/\pi) \int_0^A \cos \lambda x F(x) dx$$

and let

$$\Phi(x, y, t) = \int_0^\infty e^{-\lambda y} \lambda^{-\frac{1}{2}} \sin \lambda^{\frac{1}{2}} t \cos \lambda x f(\lambda) d\lambda.$$

Then Φ is the velocity potential for the plane motion of a heavy fluid initially at rest and occupying the region $y \geqq F(x)$ (y-axis in the direction of gravity) when the linearized free surface condition is assumed. [This may be considered as a generalization of formula (7), p. 384 of Lamb's Hydrodynamics [6th ed., Cambridge, 1932] which is applicable only to a more restricted class of initial free surface shapes.] *J. V. Wehausen* (Providence, R. I.).

Kotchin, N. E. On the wave-making resistance and lift of bodies submerged in water. The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No. 1-8, ii+125 pp. (1 plate) (1951).

Translated from the Transactions of the Conference on the Theory of Wave Resistance, Central. Aero-Gidrodinam. Inst., Moscow, 1937, pp. 65–134.

Kotchin, N. E. The two-dimensional problem of the steady oscillations of bodies under the free surface of a heavy incompressible liquid. The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No. 1-9, ii+40 pp. (1952).

Translated from Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1939, 37–62.

Kotchin, N. E. The theory of waves generated by oscillations of a body under the free surface of a heavy incompressible fluid. The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No. 1-10, 39 pp. (1952).

Translated from Učenye Zapiski Moskov. Gos. Univ. Mehanika 46, 85–106 (1940); these Rev. 12, 59.

Haskind, M. D. Translational motion of bodies under the free surface of a heavy fluid of finite depth. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1345, 20 pp. (1952).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 9, 67–78 (1945); these Rev. 8, 235.

Stewartson, K. On the slow motion of a sphere along the axis of a rotating fluid. Proc. Cambridge Philos. Soc. 48, 168–177 (1952).

The author considers, in an inviscid incompressible fluid rotating about an axis with uniform angular velocity, the velocity perturbations caused by the slow uniform motion of a sphere along the axis after an impulsive start. Interest in this problem was initiated by experiments of G. I. Taylor in which it was observed that a cylinder of the same diameter as the sphere seems to be pushed along in front of it. In this paper the author, assuming that the relative motion is small, obtains by the method of Laplace transforms an integral representation for the solution from which he derives asymptotic expressions for the velocity and pressure. These give a description of the ultimate motion in agreement with Taylor's experiments. In the recent work of Morgan [Proc. Roy. Soc. London. Ser. A. 206, 108–130 (1951); these Rev. 13, 81] who also considered this problem, there is an account of the previous theoretical work on the subject. *D. Gilbarg* (Bloomington, Ind.).

Casal, Pierre. Sur l'énergie cinétique d'un écoulement possédant une surface de discontinuité de vitesse. C. R. Acad. Sci. Paris **234**, 804–806 (1952).

The author shows that among three-dimensional flows with slip surface of discontinuity a flow making the kinetic energy stationary with respect to all flows with a neighboring discontinuity surface is irrotational and that the velocity magnitude and therefore the pressure is continuous across the slip surface. This is related to a result of Riabouchinsky [same C. R. **185**, 840–841 (1927)] on minimum virtual mass.

D. Gilbarg (Bloomington, Ind.).

Lord, W. T. Free-streamline jets in shear flow. Proc. Cambridge Philos. Soc. **48**, 197–201 (1952).

This note considers the plane flow of a jet from an aperture in a wall bounding a semi-infinite fluid, but unlike the classical irrotational flow problem which is easily treated by the method of Kirchhoff, the flow is here required to have constant non-zero vorticity. For small values of the vorticity σ the author proposes to determine the stream function as a perturbation in powers of σ of the basic irrotational flow. He formulates the boundary value problem satisfied by the first perturbation and finds expressions in terms of it for the shape of the jet to the same approximation. *D. Gilbarg.*

Brard, R. Cas d'équivalence entre carènes et distributions de sources et de puits. Bull. Assoc. Tech. Maritime Aéro. no. **49**, 189–220; discussion, 221–230 (1950).

The author considers bodies generated by source-sink distributions, the hydrodynamic force on such a body, and the limitations of Lagally's theorem in computing this force. The author's use of the vocabulary of measure theory seems hardly justified by the use to which it is put.

J. V. Wehausen (Providence, R. I.).

***Streeter, V. L. The ring doublet in ideal fluid flow.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 56–65 (1 plate). J. W. Edwards, Ann Arbor, Michigan, 1951.

The doublet ring in nonviscous, incompressible flow is a uniform distribution of infinitesimal point doublets around a circle, with all doublet axes in the same direction and perpendicular to the plane of the ring. Taking the ring in the (y, z)-plane with center at the origin and with doublet axes in the $+x$-direction, the flow has axial symmetry about the x-axis. The velocity components, velocity potential, and the Stokes stream function are computed for this flow, in an integrated form, starting with the velocity potential of a point doublet. *Extract from the paper.*

Cunsolo, Dante. I profili di Joukowski a punta arrotondata. Aerotecnica **32**, 20–24 (1952).

A circle C which passes through $\zeta=-c$ and contains $\zeta=c$ in its interior is mapped by $z=x+iy=\zeta+c^2/\zeta$ into a Joukowski airfoil J with cusped trailing edge at $z=-2c$. The author shows that replacing C by a concentric circle C' with slightly larger suitably chosen radius will yield a profile which closely resembles J near its leading edge but has prescribed inclination between the upper and lower surfaces near the trailing edge, the curvature of which is so great as to make it practically pointed. The radius and center of C' for such an airfoil of prescribed chord, thickness, and inclinations between the x-axis and the upper and lower surfaces at the trailing edge can easily be found by means of approximate formulae given in the paper. *J. H. Giese.*

Smirnov, A. I. On the determination of the circulation and lifting force of an arbitrary thin wing located near to a wall. Akad. Nauk SSSR. Inženernyĭ Sbornik **9**, 45–56 (1951). (Russian)

Thin wing theory has been applied to the flow past an infinite wing of small curvature and its mirror image with respect to a plane wall. The boundary condition at the airfoil yields an integral equation

$$(*) \qquad 2\pi V_\infty \sin \alpha_{0_k} + \int_0^b I(x, \bar{x}, x_k)\gamma(x)(x-x_k)^{-1}dx = 0$$

for the vortex strength $\gamma(x)$ at $(x, 0)$ on the chord, where $(0, 0)$ is at the leading edge, $(b, 0)$ at the trailing edge, α_{0_k} is the local angle of attack at $(x_k, 0)$, and $I(x, \bar{x}, x_k)$ is an easily determined function of x_k, x, and the mirror image $(\bar{x}, \bar{y})$ of $(x, 0)$ with respect to the wall. The author sets $\gamma(x)I(x, x, x_k)/2V_\infty = B_{k0}\cot\frac{1}{2}\theta + \sum_1^m B_{kn}\sin n\theta$, and $\gamma(x)/2V_\infty = A_0\cot\frac{1}{2}\theta + \sum_1^m A_n \sin n\theta$, where $x = \frac{1}{2}b(1-\cos\theta)$. Then for $\theta_k = k\pi/(m+1)$, $0 \leqq k \leqq m$, (*) is imposed at $x_k = x(\theta_k)$ to obtain $m+1$ linear equations for A_n, $\gamma(x_k)$, from which the circulation $\Gamma = bV_\infty(A_0+\frac{1}{2}A_1)\sin\alpha$ can be found. In calculating the lift $Y = \rho\int_0^b V\gamma(x)dx$, V must be taken to be the component of velocity at $(x, 0)$ parallel to the wall of the resultant of the velocity V_∞ of the main flow and the velocity induced by the image airfoil. For lift computation the author replaces the vortex distribution $\gamma(x)$ by discrete vortices of strength

$$\Gamma_k = \int_{x_k}^{x_{k+1}} \gamma(x)dx \quad \text{at} \quad x_k = (k-\tfrac{1}{2})b/\nu,\ 1 \leqq k \leqq \nu,\ x_{\nu+1} = b,$$

and $-\Gamma_k$ at their images. His computations with $m=3$ and $\nu=1$ for a flat plate lead to excellent agreement with exact values of Γ and Y as functions of angle of attack α and relative distance H/b from the wall. Γ and Y have also been computed for circular arc airfoils as functions of curvature, α, and H/b. Finally, for $\alpha \leqq 10°$ close agreement has been found between calculated and experimental lift coefficients for symmetrically mounted pairs of NACA four-digit series airfoils. *J. H. Giese* (Havre de Grace, Md.).

Neumark, S. Pressure distribution on an airfoil in non-uniform motion. J. Aeronaut. Sci. **19**, 214–215 (1952).

Söhngen, H. Durchgang einer Potentialstörung durch einen Leitschaufelkranz. Ing.-Arch. **20**, 13–18 (1952).

L'auteur étudie la propagation à travers les aubes d'une roue fixe (distributeur ou diffuseur) de la perturbation due à une série de nervures de soutien. Le fluide est supposé parfait et incompressible et le mouvement est potentiel. La hauteur du canal annulaire est petite par rapport au rayon de sorte que le mouvement peut être considéré comme plan. L'intervalle entre les aubes est supposé petit par rapport à leur longueur. Les deux cas suivants sont examinés: 1) les nervures sont placées devant le distributeur; l'auteur montre que dans ce cas la perturbation ne traverse pas le distributeur; 2) les nervures sont placées derrière le diffuseur; dans ce cas la perturbation due aux nervures n'est pas supprimée par la présence du diffuseur, mais elle est au contraire renforcée. *R. Berker* (Istanbul).

Railly, J. W. The flow of an incompressible fluid through an axial turbo-machine with any number of rows. Aeronaut. Quart. **3**, 133–144 (1951).

The author considers approximate axially symmetric solutions for the problem of flow through an axial turbomachine

having cylindrical boundaries, such that either the distribution of angular momentum or the blade outlet angles are prescribed. The radial velocity u is taken to be of the form $u(r, z) = e^{\pm kz}f(r)$ where r, z are the radial and axial coordinates respectively, $f(r)$ is a function to be determined and k is an unknown constant. The axial velocity $w(r, z)$ follows immediately from the continuity relation and after boundary conditions are applied, the value of $f(r)$ follows from a radial equilibrium consideration at $z=\infty$ and only the value of k remains to be determined. This is done by adjusting k to satisfy the equations of motion, in the mean, over the region of flow. The author considers solutions of problems involving many blade rows by building them from solutions of this type. Detailed calculations are made for the machine with an infinite succession of similarly shaped stages [rotor-stator combinations] and the ultimate repetitive pattern is computed. *F. E. Marble* (Pasadena, Calif.).

Wu, Chung-Hua. A general theory of three-dimensional flow in subsonic and supersonic turbomachines of axial-, radial-, and mixed-flow types. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2604**, ii+93 pp. (1952).

This paper, one of a group written on this subject by the same author [see, e.g., same Tech. Notes, no. **2407** (1951); these Rev. **13**, 164], discusses in detail a method for setting up the problem of flow of a gas through a rather general turbomachine with finite number of blades, so that appropriate relaxation or characteristic numerical calculations may be systematically carried out. The device employed to make the calculation tractable is to consider the flow on two sets of coordinate surfaces, one made up of nearly radial elements, the other consisting of elements nearly circular around the turbomachine axis. The two surfaces are actually composed of stream tubes and are determined to be respectively radial and circular at some appropriate point along the turbomachine axis. The aim of this technique seems to be one of providing a new coordinate system such that the flow is relatively independent on the two sets of coordinate surfaces and hence involves only weakly coupled pairs of two-dimensional problems.

The mathematical problem is then that of determining the quasi-two-dimensional flow in each of these two sets of surfaces and the shapes of these surfaces satisfying the wall boundary conditions and whatever information the problem at hand provides concerning the blades. The author writes the equations of continuity, motion, energy, etc. for each set of surfaces and, obviously, they are coupled. Likewise the surfaces are twisted due to the three-dimensionality of the flow. The problem is then numerically solved by making an assumption [for example] as to the shape of the first set of surfaces and calculating [numerically] the two-dimensional flow in them, estimating the coupling terms with the flow in the second set of surfaces. With this result, the shape of the second set of surfaces may be estimated, and the flow in these surfaces computed calculating the coupling terms from the flow in the first set. The process may be iterated, the detailed technique depending considerably upon the problem at hand.

The idea of the method is simple and straightforward; the labor involved in arranging the equations and boundary conditions in a form so that this numerical program may be carried out systematically is very great and this result constitutes the main contribution of the paper. The calculation for a particular example still appears to be quite lengthy, somewhat out of range of a desk calculator. The procedure does, however, allow the possibility of computing the details of flow for problems of special significance so that the important new features of the flow may be observed.

F. E. Marble (Pasadena, Calif.).

Meixner, J. Strömungen von fluiden Medien mit inneren Umwandlungen und Druckviskosität. Z. Physik **131**, 456–469 (1952).

The author claims that fluids do not have a "proper pressure viscosity" because "one could hardly find a mechanism" for it. He states, however, that a "relaxation process" gives rise to an apparent pressure viscosity. By a relaxation process he means diffusion in a two-component mixture; the components need not be restricted to any particular physical category: they may be different substances which may undergo chemical reactions, different energy states of the same substance, or any quantities obeying simple laws of conservation and combination. Then, supposing there exist a caloric equation of state $U=U(S, V, \xi, 1-\xi)$, where S is the specific entropy, V the specific volume, ξ the specific concentration of one of the two components, we have $dU=TdS-PdV-Ad\xi$, where $A=\mu_2-\mu_1$ is the difference of the chemical potentials μ_2 and μ_1 of the components whose concentrations are ξ and $1-\xi$. To this is added the assumption $A=-\alpha(S, V)[\xi-\bar{\xi}(S, V)]$, where $\bar{\xi}$ is the assumed root of $A(S, V, \xi)=0$, and thus is the equilibrium value of ξ, and where α is a positive function. These assumptions are customary in what has come to be called "irreversible thermodynamics." Next the author proposes $\rho d\xi/dt=\epsilon(S, V)A(S, V, \xi)$, where ϵ is a positive function. Hence

$$(*)\qquad \frac{d\xi}{dt}=-\frac{1}{\tau}[\xi-\bar{\xi}(S, V)],$$

where the "relaxation time" τ is defined by $\tau=\rho/(\epsilon\alpha)$, and hence $\tau=\tau(S, V)$.

The remainder of the paper is devoted to approximate integration of (*) for "slow" processes and of its generalization to the case of two relaxation processes. The author's result for a single process is that there is an effective pressure viscosity $\zeta=(\partial\bar{\xi}/\partial V)^2/\epsilon$; for two processes he obtains a more complicated formula involving three different pressures, whose interpretation he says is not simple, although he says they furnish a non-trivial example of the Onsager-Casimir reciprocity relations.

The reviewer is unable to find the introduction of so many phenomenological assumptions in order to obtain a "mechanism" for bulk viscosity more convincing than the usual method of continuum mechanics. Moreover, to obtain a "relaxation time" for acoustical phenomena such assumptions are quite unnecessary, since absorption resonance at a specific frequency is a consequence of the classical absorption formula given by Stefan [S.-B. Akad. Wiss. Wien **53**, Abt. 2, 529–537 (1866)] (cf. also Gurevich [C. R. (Doklady) Acad. Sci. URSS **55**, 17–19 (1947)], C. Eckart [Physical Rev. **71**, 277 (1947)]). *C. Truesdell.*

Gotusso, Guido. Sopra un principio variazionale nei liquidi viscosi. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 380–388 (1951).

A variational principle for slow motions or for motions in which $w\times v=0$ for viscous incompressible fluids is obtained. The quantity varied is the Laplacian of the kinetic energy.

C. Truesdell (Bloomington, Ind.).

van Deemter, J. J. Bernoulli's theorem for viscous fluids. Physical Rev. (2) 85, 1049 (1952).

By eliminating the dissipation function between the thermodynamic energy equation and the energy dissipation formula which follows from the momentum equation, the author obtains

$$\frac{\partial}{\partial t}[\rho(\tfrac{1}{2}\mathbf{v}^2+u)]-\rho\mathbf{v}\cdot\mathbf{f}$$
$$+\nabla\cdot\left[\rho\mathbf{v}\left(\tfrac{1}{2}\mathbf{v}^2+u+\frac{p}{\rho}\right)-2\mu\mathbf{v}\cdot\operatorname{def}\mathbf{v}-\lambda\mathbf{v}\nabla\cdot\mathbf{v}+\mathbf{q}\right]=0,$$

where the notations are classical. He states that for steady flow of a perfect gas this result can be applied to the theory of the Ranque-Hilsch vortex tube. [The reviewer does not believe that the distinction made by the author between two methods of obtaining Bernoulli theorems is material; the reason the author obtains a basically different result from the reviewer's [same Rev. 77, 535–536 (1950); these Rev. 11, 472] is that he uses the principle of conservation of energy, while the reviewer's theorem is purely dynamical. The author's result is virtually the same as eq. (E'') of A. Vazsonyi [Quart. Appl. Math. 3, 29–37 (1945); these Rev. 7, 226].] *C. Truesdell* (Bloomington, Ind.).

Homann, F. The effect of high viscosity on the flow around a cylinder and around a sphere. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1334, 29 pp. (1952).

Translated from Z. Angew. Math. Mech. 16, 153–164 (1936).

Dizioğlu, Bekir. Die mittleren Temperaturen in Schmierschichten zwischen parallelen wärmeundurchlässigen Wänden. Rev. Fac. Sci. Univ. Istanbul (A) 17, 61–65 (1952). (German. Turkish summary)

This paper deals with the problem of finding the temperature distribution in a steady one-dimensional flow of incompressible viscous fluid under pressure between two parallel and heat-insulated plates, moving relatively with a constant velocity. By neglecting the conduction of heat in the downstream direction, the problem is shown to be reducible to a boundary-value problem of the Sturm-Liouville type. From the orthogonal relations, the author deduces that the heat carried by convection at any section is equal to the total heat dissipated in the volume bounded by that section. The consequence of this is that the average temperature defined in terms of the convected heat increases linearly as the distance from the initial section in the downstream direction. It might be remarked that this result can be simply obtained by integrating the energy equation without knowing the temperature distribution. *Y. H. Kuo* (Ithaca, N. Y.).

Corrsin, Stanley. Generalization of a problem of Rayleigh. Quart. Appl. Math. 10, 186–189 (1952).

C'est l'étude mathématique du mouvement engendré au sein d'un fluide visqueux et compressible par la mise en mouvement d'un plan indéfini, parallèlement à lui même. Les inconnues, vitesse, température, pression, coefficient de frottement, sont explicitées en fonction de la variable spatiale et du temps. *R. Gerber* (Grenoble).

***Tifford, Arthur N. On certain particular solutions of the laminar boundary-layer equations.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 81–90. J. W. Edwards, Ann Arbor, Michigan, 1951.

By a general similarity transformation, it is shown that, aside from Falkner and Skan's family of boundary-layer profiles, there are two other types of main-stream velocity distributions for which the velocity profiles are similar, for walls with and without suction. Of these one is a generalized Falkner-Skan form and the other is exponentially increasing or decreasing in the stream direction. *Y. H. Kuo.*

***Oudart, Adalbert. Calcul de la couche limite compressible. Applications pratiques élémentaires des méthodes Oswatitsch-Walz et Gruschwitz-Walz.** Publ. Sci. Tech. Ministère de l'Air, no. 258, Paris, 1952. iv+66 pp. 550 francs.

This is a continuation of the author's previous report [same Publ., no. 223 (1949); these Rev. 12, 552] and presents a collection of formulae and procedures for the practical calculation of laminar and turbulent boundary layers of compressible fluids, based on the methods of Oswatitsch and Walz and Gruschwitz and Walz. The cases considered are flows over axially symmetric cones and fins on projectiles. *Y. H. Kuo* (Ithaca, N. Y.).

Morgan, G. W. On the steady laminar flow of a viscous incompressible fluid in an elastic tube. Bull. Math. Biophys. 14, 19–26 (1952).

This paper presents a systematic method of approximation to the solution of steady flow of a viscous incompressible fluid through an elastic tube. If the Poiseuille flow is taken as the first approximation, the author shows that a consistent approximation results if the product of the Reynolds number $\rho VR/\mu$ and a parameter $\mu V/\delta E$ as well as $\mu V/\delta E$ are small; here ρ, μ, V, R, δ, and E denote respectively density, viscosity coefficient, mean velocity of the fluid, radius, thickness, and Young's modulus of the tube. Consequently, a second approximation is obtained for either $\rho VR/\mu\gg1$ $(\rho VR/\mu)(\mu V/\delta E)\ll1$ and $\rho VR/\mu=O(1)$, $\mu V/\delta E\ll1$; or $\rho VR/\mu\ll1$ and $\mu V/\delta E=O(\rho VR/\mu)$. *Y. H. Kuo.*

Dunn, D. W., and Lin, C. C. The stability of the laminar boundary layer in a compressible fluid for the case of three-dimensional disturbances. J. Aeronaut. Sci. 19, 491 (1952).

Lessen, Martin. Note on a sufficient condition for the stability of general, plane parallel flows. Quart. Appl. Math. 10, 184–186 (1952).

En généralisant les résultats de Synge [Amer. Math. Soc. Semicentennial Publ., New York, 1938, pp. 227–269], l'auteur donne des conditions suffisantes de stabilité pour les mouvements plans, parallèles les plus généraux d'un liquide réel. *R. Gerber* (Grenoble).

Lessen, Martin. Some considerations of the stability of laminar parallel flows. J. Aeronaut. Sci. 19, 492 (1952).

Mattioli, E. La teoria statistica della turbolenza. Aerotecnica 32, 25–42, 87–98 (1952).

A comprehensive survey of modern developments in the statistical theory of turbulence.

Aržanyh, I. S. Functions of the stress tensor of hydrodynamics. Doklady Akad. Nauk SSSR (N.S.) 83, 195–198 (1952). (Russian)

Hydrodynamic analogs of Maxwell's and Morera's stress functions in elasticity can be found as follows. Write the hydrodynamic equations (1) $\partial\rho/\partial t+\operatorname{div}(\rho\mathbf{v})=0$, (2) $\partial(\rho\mathbf{v})/\partial t+\operatorname{div}(\rho\mathbf{v}\mathbf{v}-P)=0$, where ρ is the density, $\mathbf{v}$ the

The title of the book was changed in page proof. It should read: "On an extension of Riemann's method of integration with application to one-dimensional gas dynamics."

velocity, P the stress tensor, and

$$\operatorname{div}(\rho\mathbf{v}\mathbf{v})=\rho\mathbf{v}\cdot\operatorname{grad}\mathbf{v}+\mathbf{v}\operatorname{div}(\rho\mathbf{v}).$$

By (1) $\rho=\operatorname{div}\mathbf{a}_1$, $\rho\mathbf{v}=-\partial\mathbf{a}_1/\partial t+\operatorname{curl}\mathbf{a}_2$, for some vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, and similarly each component of $\rho\mathbf{v}$ and each row of $\rho\mathbf{v}\mathbf{v}-P$ is expressible in terms of div and curl of vectors $\mathbf{a}_3, \cdots, \mathbf{a}_8$. The two forms for $\rho\mathbf{v}$ and the symmetry of $\rho\mathbf{v}\mathbf{v}-P$ yield six more equations of the form $\partial b/\partial t+\operatorname{div}\mathbf{c}=0$ involving the twenty-four components of $\mathbf{a}_1, \cdots, \mathbf{a}_8$, which are now expressible in terms of div and curl of new vectors $\mathbf{A}_1, \cdots, \mathbf{A}_{12}$. Eventually ρ, $\rho\mathbf{v}$, and $\rho\mathbf{v}\mathbf{v}-P$ can be expressed as linear combinations, with coefficients ±1 or 0, of second partial derivatives of twenty-one independent linear combinations of components of $A_1, \cdots, A_{12}$. Conversely, if these twenty-one hydrodynamic "stress" functions are chosen arbitrarily of class C_2 in t, x, y, z, then ρ, $\rho\mathbf{v}$, and P defined with their aid satisfy (1) and (2). *J. H. Giese.*

Giese, J. H. Stream functions for three-dimensional flows. J. Math. Physics 30, 31–35 (1951).

On sait que l'écoulement adiabatique le plus général d'un fluide parfait, compressible, peut être représenté au moyen de deux fonctions de courant. L'auteur montre que ces fonctions jouissent de propriétés simples, analogues à celles de la fonction de courant des mouvements plans, et il construit une expression intégrale de ces deux fonctions qui permet de retrouver les équations du mouvement par un principe variationnel. *R. Gerber* (Grenoble).

Shiffman, Max. On the existence of subsonic flows of a compressible fluid. Proc. Nat. Acad. Sci. U. S. A. 38, 434–438 (1952).

The problem considered is that of steady, irrotational, plane flow of a compressible fluid past a fixed smooth object. The adiabatic law $p=A\rho^\gamma$, $\gamma>1$, is assumed, and flow velocity and Mach number M_0 are prescribed at infinity. The following result is proved: there is a number $\hat{M}_0<1$ such that for any $M_0<\hat{M}_0$ there exists a uniquely determined completely subsonic flow solving the stated problem; this flow depends continuously on M_0. The existence proof, which in part is only sketched, rests on the formulation of a double integral variational problem of known type for the stream function. Known results of variational calculus are then applied. No proof is given for the uniqueness of the flow, but this can be shown directly from the partial differential equation for the stream function. *J. B. Serrin.*

Sauer, Robert. Elementare Lösungen der Wellengleichung isentropischer Gasströmungen. Z. Angew. Math. Mech. 31, 339–343 (1951). (German. Russian summary)

It is well known that the introduction of Riemann's characteristic variables reduces the equations for nonstationary isentropic one-dimensional flow to a single linear equation of the second order which, for polytropic gases $p=\text{const.}\times\rho^\kappa$, $\kappa=(2n+3)/(2n+1)$, becomes the Darboux equation, $f_{\lambda\mu}+n(\lambda+\mu)^{-1}(f_\lambda+f_\mu)=0$. In this paper the author determines all pressure-density relations for which the flow equations lead to the Darboux equation, carrying this out for both the non-stationary one-dimensional flows and the steady plane irrotational flows. The cases $n=0$ and $n=-1$ are discussed in detail. *D. Gilbarg.*

Ludford, Geoffrey S. S. On multiple coverings of the speedgraph plane in the one-dimensional unsteady motion of a perfect gas. Proc. Cambridge Philos. Soc. 48, 499–510 (1952).

The initial value problem for one-dimensional isentropic unsteady motion of a perfect gas may be formulated as a Cauchy problem for a linear hyperbolic partial differential equation [Courant and Friedrichs, Supersonic flow and shock waves, Interscience, New York, 1948, p. 89; these Rev. 10, 637]. This paper investigates flows whose initial values ($u=u(x)$, $a=a(x)$ at $t=0$) are such that the initial curve for the Cauchy problem is tangential at some point to a characteristic. Although such a Cauchy problem has no continuous single-valued solution, the author succeeds in defining a physically meaningful, multiple-valued solution, and obtains a neat integral representation for it. This not only resolves a difficulty in the classical treatment of the initial value problem, but also simplifies consideration of the flow problem. For flow in an infinite tube it is shown that initial data of the above sort guarantee the occurrence of a limit line (and hence eventual breakdown of the flow); for flow in a closed tube it is shown that, no matter what (nonconstant) initial data are prescribed, the flow will break down. *J. B. Serrin* (Princeton, N. J.).

Grib, A. A. Integration of the equations of unsteady motion of a liquid for hydraulic shock in long conduits. Doklady Akad. Nauk SSSR (N.S.) 83, 43–46 (1952). (Russian)

Under the usual hydraulic assumptions the equations for unsteady, nonviscous, slightly compressible flow in an elastic tube of circular cross-section can be written in the linearized, one-dimensional form $F_0\partial H/\partial t=-a^2\partial Q/\partial s$, $F_0\partial H/\partial s=-\partial Q/\partial t$, with $H=gz+p/\rho_0$, and

$$a^2=K\rho_0^{-1}/(1+KD_0/lE).$$

Here Q is the rate of volume flow, s a longitudinal coordinate, $D_0(s)$ the inner diameter of the pipe, $F_0(s)=\frac{1}{4}\pi D_0^2$, $z(s)$ the elevation of its center line, $l(s)$ the thickness of its wall, E the Young's modulus for the wall, and K the bulk modulus of elasticity of the liquid. Others have solved this system numerically by the method of characteristics, but the author suggests that for qualitative purposes an analytical approximation may be preferable. Let $T=\int_0^s ds/a$ and introduce characteristic variables $2\xi=T-t$, $2\eta=T+t$ to obtain (1) $F_0\partial H/\partial\eta=-a\partial Q/\partial\eta$, (2) $F_0\partial H/\partial\xi=a\partial Q/\partial\xi$. Approximate integrals can be found by the method previously applied to steady plane flow by S. A. Christianovich [Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 215–222 (1947); these Rev. 9, 390]. Approximate a/F_0 by segments of curves $f(T)=A(T+C)^{2n}$, where A, C, and n are constants. Then (1) and (2) imply the Euler-Darboux equation

$$(\xi+\eta)\partial^2Q/\partial\xi\partial\eta+n(\partial Q/\partial\xi+\partial Q/\partial\eta)=0.$$

If n is an integer there are simple closed forms for Q and H involving an arbitrary function of ξ and another of η only. Explicit forms are exhibited for $n=1$ (-1) for conical (hyperboloidal) pipe sections. *J. H. Giese.*

Jones, Doris M., Martin, P. Moira E., and Thornhill, C. K. A note on the pseudo-stationary flow behind a strong shock diffracted or reflected at a corner. Proc. Roy. Soc. London. Ser. A. 209, 238–248 (1951).

In this note the equations of an unsteady compressible flow in the (x, y)-plane, which is expressible in terms of x/t and y/t only, are transformed into those of a steady com-

pressible flow with a non-conservative field of external forces and a field of sinks. The steady-flow problems of this type, which correspond to shock reflexion and diffraction at a corner, are then discussed qualitatively. It is shown that, under certain conditions, there are regions in the corresponding steady flows which are entirely supersonic and for which a simple solution can be given without determining the whole field of flow. No complete solution for the whole field of flow has yet been given. (From the author's summary.) *J. B. Serrin* (Princeton, N. J.).

Borg, S. F. On unsteady nonlinearized conical flow. J. Aeronaut. Sci. 19, 85–92, 100 (1952).

This is a discussion of the equations of plane motion of an ideal compressible fluid for cases where the field grows in proportion to time and does not involve a characteristic length parameter. Thus, the flow is "conical" in an (x, y, t)-space. The specific case considered is that of a plane shock wave striking an infinite wedge, sometimes called "shock diffraction." The problem has previously been treated in the linearized approximation, i.e., for small wedge angles. First, the equations of motion are written down for the case described above, and the invariance in a conical coordinate system implied there is verified. It is then proved that a two-shock reflection pattern with attachment at the origin is impossible for $\gamma=7/5$. A reciprocity between two very unlike flow patterns, one involving two-shock reflection with detachment before the origin, the other three-shock reflection with attachment, is pointed out. It is not known whether either of these exists. Finally, the presence of certain special characteristic curves is discussed, and some conclusions are drawn regarding existence of solutions. For a similar discussion of the same problem see the paper reviewed above. *W. R. Sears* (Ithaca, N. Y.).

Moeckel, W. E. Interaction of oblique shock waves with regions of variable pressure, entropy, and energy. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2725**, 34 pp. (1952).

Equations are derived for computing the form of an oblique shock wave as it passes through supersonic regions in which static pressure, stagnation pressure, stagnation temperature, or combinations of these are continuously variable. Rigorous portions of the analysis are limited to shock strengths for which the flow downstream of the shock remains supersonic. When no downstream waves other than those generated by the interaction process are present, the rate of change of shock angle with upstream Mach number is found to be a function only of the local shock angle and upstream Mach number; hence, the propagation through a nonuniform region depends only on the initial shock strength and Mach number. A procedure is described for computing the supersonic portion of the flow field downstream of the shock wave.

For the special cases of supersonic shear flow and Prandtl-Meyer flow, charts of the shock angle as a function of upstream Mach number are presented so that the passage of a shock wave through these types of nonuniform regions can be easily traced. For a prescribed initial shock strength and initial shock strength and initial Mach number, a minimum upstream Mach number is found below which no physically realistic solution can be obtained with the equations for simple propagation. This result serves as a sufficient condition for the avoidance of separated flow, reversed flow, or other upstream effects. An example is computed of the propagation of a shock wave through a wake-type supersonic shear profile and the flow field downstream of a shock is constructed. (Author's summary.) *Y. H. Kuo.*

Shu, S. S. On two-dimensional flow after a curved stationary shock (with special reference to the problem of detached shock waves). Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2364**, 19 pp. (1951).

C'est l'étude mathématique de l'écoulement plan d'un gaz parfait en aval d'une onde de choc incurvée. La forme de l'onde est supposée donnée, ce qui détermine certaines conditions initiales des variables, et l'écoulement se trouve alors être défini par un problème aux limites de Cauchy. Une approximation analytique de la fonction de courant est obtenue pour la région subsonique en aval de l'onde; la densité est alors calculée avec l'équation de Bernouilli à partir de la fonction de courant. Enfin la ligne sonique est déterminée à l'aide du champ des vitesses. *R. Gerber.*

Busemann, Adolf. Application of transonic similarity. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2687**, 22 pp. (1952).

From a review of the different similarity approaches to compressible potential flow, the meaning and limitations of transonic similarity are traced back to their origin. Although the main text deals with the quasi-two-dimensional flow, special suggestions for the case of axisymmetrical bodies are added in an appendix. (Author's summary.) *Y. H. Kuo.*

Pai, S. I. Axially symmetrical jet mixing of a compressible fluid. Quart. Appl. Math. 10, 141–148 (1952).

The author considers the steady flow of an axially symmetric jet of a compressible fluid. For laminar flow, the problem is solved both approximately by assuming the flow inside the jet differing slightly from that of the surrounding stream and exactly for the Prandtl number unity and viscosity varying with temperature T according to $T^{0.76}$. For turbulent flow, if the exchange coefficient assumes a simple power law, the mathematical problem is shown reducible to that of the previous laminar case.
Y. H. Kuo (Ithaca, N. Y.).

Toose, D. G. The laminar motion of a plane symmetrical jet of compressible fluid. Quart. J. Mech. Appl. Math. 5, 155–164 (1952).

This paper deals with the problem of a compressible symmetrical plane jet issuing from a slit by assuming the Prandtl number to be unity and the viscosity varying linearly with temperature. For this case, the problem is transformed to that of an incompressible case as previously shown by Howarth [Proc. Roy. Soc. London. Ser. A. 194, 16–42 (1948); these Rev. 10, 270]. As numerical examples, both compressibility and temperature effects are studied.
Y. H. Kuo (Ithaca, N. Y.).

***Hall, Newman A. The action of friction in nonsteady flow of fluids.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 340–353. J. W. Edwards, Ann Arbor, Michigan, 1951.

This paper formulates the problem of one-dimensional unsteady flow of gases with wall friction by the method of characteristics. A form for the friction factor entering the momentum equation is suggested, but its justification seems difficult. *Y. H. Kuo* (Ithaca, N. Y.).

Shen, S. F. On the boundary-layer equations in hypersonic flow. J. Aeronaut. Sci. 19, 500–501 (1952).

Westley, R. The potential due to a source moving through a compressible fluid and applications to some rotary derivatives of an aerofoil. Coll. Aeronaut. Cranfield. Rep. no. 54, 26 pp. (6 plates) (1952).

The first part of this note concerns the evaluation of the potential at a fixed point in space due to an arbitrarily moving source. The method is then applied to the calculation of the disturbances at a point fixed relative to a source moving in a helical path where conditions are invariant with time. An explicit relation for the potential is obtained if the rate of rotation is assumed small, and the results are applied to the calculation of the pressure distribution on a wing in a uniform rotary motion in yaw at supersonic speeds. The quasi-static yawing derivative of the rolling moment is then calculated for an infinite-aspect-ratio wing. It is found that the curvature of the path of the wing must be taken into account, except in the particular case of zero sweepback of the wing leading edge. Below a certain supersonic Mach number the rolling moment is unstable, and this effect is most pronounced for high sweepback. (Author's summary.)

Y. H. Kuo (Ithaca, N. Y.).

Lance, G. N. The drag on slender pointed bodies in supersonic flow. Quart. J. Mech. Appl. Math. 5, 165–177 (1952).

Ward [same J. 2, 75–97 (1949); these Rev. 10, 644] showed that von Kármán's well-known integral formula for the pressure drag of slender bodies of revolution in axisymmetric supersonic flow according to the linear perturbation approximation applies as well to bodies of rather arbitrary cross-section. Von Kármán's formula is re-derived here, and two new, alternative integral formulas are obtained. The first of these involves $S''(x)$, where $S(x)$ is the distribution of cross-sectional area along the body length; the second is (except for a dimensional constant of proportionality)

$$(*)\qquad D=\frac{1}{2\pi}\int_0^\infty nQ(n)Q(-n)dn$$

where $Q(n)$ is the Fourier transform of $S'(x)$. As an example, the case $S(x)=(\pi/4)l^3\sin^3\pi x$ is worked out using (*), and a correction is made in the numerical result obtained for this case by Lighthill [Quart. J. Math., Oxford Ser. 20, 121–123 (1949); these Rev. 11, 223].

W. R. Sears.

Imai, Isao. Note on the drag of a finite wedge at Mach number 1. J. Aeronaut. Sci. 19, 496–497 (1952).

Dedecker, Paul. Sur le théorème de la circulation de V. Bjerknes et la théorie des invariants intégraux. Inst. Roy. Météorolog. Belgique. Misc. no. 36, 63 pp. (1951).

It is plain that the theory of integral invariants was inspired in part by the Helmholtz and Kelvin vorticity theorems for barotropic flow of perfect fluids. As it is usually presented, however, there is little or no connection with hydrodynamical ideas. The author gives a most valuable systematic exposition of both subjects. He begins with a brief and lucid explanation of the elements of the exterior differential calculus; this apparatus he uses to develop the theory of integral invariants in a concise and rigorous way. The author's treatment here recommends itself as a highly suitable replacement for the deluge of unexplained differentials and variations in the usual dynamics courses. The author points out the counterparts of the Lagrange, Helmholtz, and Kelvin theorems in the general theory of equations deriving from a variational principle.

Next the author takes up the simpler of the two inverse problems of the calculus of variations for the case of first derivatives only, viz., given a system of equation $F_a(t, q^j, \dot{q}^j, \ddot{q}^j)=0$ $(a, j, =1, \cdots, n)$, does there exist a function $L=L(t, q^i, \dot{q}^i)$, one of whose variational derivatives is identical with each F_a? This problem was given an affirmative answer by Bateman [Physical Rev. 38, 815–819 (1931)], who put $L=s^aF_a$, where the s_a are n new unknowns, to be eliminated by means of the n additional equations obtained by annulling the variational derivatives; thus the complete set of extremals are curves in a $2n$-dimensional space, whose projections onto the original n-dimensional space yield the desired trajectories. The author discusses the invariance of Bateman's result. Then he applies the method to the hydrodynamics of perfect fluids. He obtains $L=g_{ij}\dot{x}^i\dot{s}^j+K_is^i$, where $\dot{x}^i$ is the fluid velocity and K_i the extraneous force. By assigning suitable initial conditions to the auxiliary field y^i, he shows that the invariant integral in the 6-dimensional space when projected onto the 3-dimensional physical space yields the Kelvin theorem in the barotropic case, the Bjerknes theorem in the baroclinic case. In course of the proof he puts the hydrodynamical equations into 12- dimensional Hamiltonian form. He adds a long and involved derivation of the condition for permanent vortex-lines. In an appendix he derives a variational principle for slow motions of viscous compressible fluids with an assigned density and pressure field, generalizing a result of Kravtchenko [C. R. Acad. Sci. Paris 213, 977–980 (1941); these Rev. 5, 192]. [The reviewer believes that by adding the energy equation and the continuity equation one should be able to apply Bateman's method successfully to the general theory of viscous compressible fluids, without foreknowledge of pressure and density or restriction to slow motions.]

C. Truesdell (Bloomington, Ind.).

Kuo, Hsiao-Lan. Dynamical aspects of the general circulation and the stability of zonal flow. Tellus 3, 268–284 (1952).

L'auteur établit l'équation linéarisée du tourbillon d'un courant zonal variable, en supposant le mouvement horizontal non divergent de densité constante et en négligeant les forces de frottement. Comme il s'agit d'un mouvement à grande échelle, on peut remplacer en première approximation les variables par leurs moyennes le long du cercle de latitude. Une fois déduit l'équation du tourbillon linéarisé, l'auteur étudie l'effet des perturbations, en représentant la fonction des vitesses perturbées comme une somme d'harmoniques simples de la forme $\psi(\gamma)e^{in(\lambda-\nu t)}$. La fonction $\psi(\gamma)$ satisfait à une équation différentielle linéaire du second ordre. Suivant la grandeur de la vitesse de propagation on trouve trois groupes différents de perturbations que l'auteur étudie en détail.

M. Kiveliovitch (Paris).

Čadež, M. Sur l'énergie potentielle du champ barique. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. 5, 5–16 (1952).

The author considers a particle of air as having a definite bounding surface and shows that, as the particle moves, there is an exchange of energy through this surface between the particle and the ambient atmosphere. A formula for the rate at which the particle receives energy from the surrounding atmosphere is established and it involves the pressure gradient and the rate at which the volume of the particle changes. Part of this energy produces a change of the potential energy of the field of pressure of the particle and the rest, a change of its internal energy. The former can

be defined in terms of the enthalpy. The author also interprets his formulae as showing that part of the energy of the particle is external to it and resides in the surrounding atmosphere. *G. C. McVittie* (Urbana, Ill.).

Markham, Jordan J. Second-order acoustic fields: relations between density and pressure. Physical Rev. (2) 86, 710–711 (1952).

The author puts the classical energy equation for inviscid non-conducting fluids (which he attributes to Eckart) into the form $\dot{p}=\kappa_s\dot{\rho}/\rho$, where κ_s is the adiabatic bulk modulus. Expanding κ_s/ρ into a series, he shows that the second-order correction to the pressure satisfies a certain partial differential equation. What he calls "the solution of interest" yields the same result as if one assumes $p=f(\rho)$ and truncates the power series after two terms. [While the author claims that his results justify the assumption $p=p(\rho)$ in the usual acoustical treatments of absorption, the reviewer is unable to see that these unjustified manipulations prove or indicate anything at all. Moreover, the assumption $p=p(\rho)$ is neither correct nor necessary in the theory of absorption.] *C. Truesdell* (Bloomington, Ind.).

Markham, Jordan J. Second-order acoustic fields: energy relations. Physical Rev. (2) 86, 712–714 (1952).

By using various power series truncated after the first one or two terms, neglecting the motion of the medium, the author derives the following formula for the time average of the stored potential energy:

$$\langle\epsilon\rangle_{\mathrm{Av}}=\frac{1}{4\rho_0^2}\left\{c^2-\frac{p_0}{\rho_0}\left[1+\frac{\rho_0}{c}\frac{\partial c}{\partial\rho}\right]\right\}R^2,$$

where R is defined by the assumed solution for the particle velocity ζ [Westervelt, J. Acoust. Soc. Amer. 22, 319–327 (1950); these Rev. 12, 140]:

$$\zeta=\frac{1}{\rho_0 k}R\sin(\omega t-kx)+\frac{R^2x}{2\rho_0^2}\left[1+\frac{\rho_0}{c}\left(\frac{\partial c}{\partial\rho}\right)\right]\cos^2(\omega t-kx).$$

The author's result contains terms not present in usual expressions. *C. Truesdell* (Bloomington, Ind.).

Mawardi, Osman K. On the generalization of the concept of impedance in acoustics. J. Acoust. Soc. Amer. 23, 571–576 (1951).

The author defines a vector impedance $\mathbf{z}$ by the equation $p=\mathbf{q}\cdot\mathbf{z}$, where p is the pressure and $\mathbf{q}$ is the particle velocity. The corresponding admittance $\mathbf{y}$ is given by $p\mathbf{y}=\mathbf{q}$. It is shown not to be inconsistent to require $(\mathbf{y}\times\mathbf{z})=0$. A Riccati-like equation is derived for $\mathbf{y}$. The acoustic impedance of a surface is defined as follows:

$$Z=\int p\bar{q}_n d\sigma\Big/S\int|q_n|^2d\sigma.$$

The value of the impedance is then obtained for a) a disk at one end of an infinitely long circularly cylindrical tube, b) a closed surface which does not enclose any sources where a low frequency limit is obtained, and c) a closed surface containing sources. *H. Feshbach* (Cambridge, Mass.).

Schoch, Arnold. Zur Frage nach dem Impuls einer Schallwelle. Z. Naturforschung 7a, 273–279 (1952).

Junger, Miguel C. Sound scattering by thin elastic shells. J. Acoust. Soc. Amer. 24, 366–373 (1952).

Chester, W. The reflection of a transient pulse by a parabolic cylinder and a paraboloid of revolution. Quart. J. Mech. Appl. Math. 5, 196–205 (1952).

The reflection of a sound pulse by a parabolic cylinder and a paraboloid of revolution is discussed. It is found that the disturbance lies wholly within a cylindrical or spherical wavefront, respectively, across which there is a discontinuity of pressure if the original pulse has a discontinuous front. At the common junction of the incident wave front, the reflected front and the barrier, the discontinuity reproduces the familiar doubling of pressure obtained in regular reflection. In other directions the discontinuity tends to zero inversely as the square root of the distance from the focus in the case of the parabolic cylinder, and inversely as the distance for the paraboloid of revolution. The excess pressure distribution along the boundary is calculated when the incident pulse is a simple step function, and the results are exhibited in tables. *A. E. Heins* (Pittsburgh, Pa.).

Elasticity, Plasticity

Isihara, Akira, Hashitsume, Natsuki, and Tatibana, Masao. Statistical theory of rubber-like elasticity. IV. Two-dimensional stretching. J. Chem. Phys. 19, 1508–1512 (1951).

Since 1936 there have been many attempts to produce a statistical theory of rubber which shows some slight agreement with experiment. The first such derivation of a formula for simple extension which agrees with the crudest approximation in elasticity theory was given by James and Guth [Physical Rev. 59, 111 (1941)]. However, this approximation was conclusively shown to be insufficient by the torsion experiment of Rivlin [J. Appl. Phys. 18, 444–449 (1947)]. The two-constant strain energy of Mooney [ibid. 11, 582–592 (1940)] is the next approximation obtained by series development of the general strain energy in elasticity theory [cf. §21 of Rivlin and Saunders, Philos. Trans. Roy. Soc. London. Ser. A. 243, 251–288 (1951)]. This approximation, which the authors call "semi-empirical," they seek to derive. After formidable calculations they succeed in getting a result equivalent to taking three terms in the series expansion for the strain energy. The expressions for the three constants in terms of molecular quantities are extremely elaborate, and the authors do not attempt to interpret them. *C. Truesdell* (Bloomington, Ind.).

Nardini, Renato. Sull'energia dissipata de forze periodiche per isteresi elastica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 371–390 (1951).

With a view to its application to the problem of energy loss occasioned by the action of the variable attraction of the sun or the moon upon the earth, the author calculates this loss for the case of a body obeying the accumulative theory of elasticity and subject to periodic forces. While a general formula has been derived by Graffi [Nuovo Cimento (8) 5, 53–71 (1928)], the author, using a symbolic method also introduced by Graffi, applies an approximate method suggested by Signorini [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 15, 151–156 (1932), and later papers]. Application is made to the case when the forces reduce to a couple. Numerical estimates for the case of the sun and moon are given. *C. Truesdell* (Bloomington, Ind.).

Danilovskaya, V. I. On a dynamical problem of thermoelasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 341–344 (1952). (Russian)

The author studies the one-dimensional dynamical problem of a semi-infinite body which is heated at the boundary plane by convection. The corresponding mathematical problem consists of a second order linear inhomogeneous partial differential equation with simple linear boundary conditions. The problem is solved for the unknown stress by an operational method which yields the solution in elementary functions. It is shown that the stress-time function at a given point consists of an analytical part and an additional part which is zero until an elastic wave reaches the point from the boundary, starting out at the instant of heating. This result is given in dimensionless form and it is shown how the stress-time curve changes its features with a change of the material thermic and elastic parameters. In all cases, after passing a minimum and a maximum, stresses decrease rapidly towards zero. *F. Niordson* (Stockholm).

Lodge, A. S. A new theorem in the classical theory of elasticity. Nature **169**, 926–927 (1952).

The author gives a transformation whereby from the solution of any problem concerning an isotropic elastic solid the solution of a corresponding problem for a certain type of anisotropic elastic solid may be deduced. The anisotropic body is supposed to possess an axis of elastic symmetry and its five elastic constants must satisfy two further requirements, expressed in Love's notations as follows:

$$\frac{c_{13}}{c_{12}}=2\frac{c_{44}}{c_{11}-c_{12}}=\left(\frac{c_{33}}{c_{11}}\right)^{\frac{1}{2}}.$$

C. Truesdell (Bloomington, Ind.).

Grioli, Giuseppe. Sulle deformazioni elastiche di un involucro omogeneo soggetto a pressione o trazione. Rend. Sem. Mat. Univ. Padova **20**, 278–285 (1951).

Schwartz's inequality is used to prove an inequality for the change of volume of a homogeneous elastic body due to the action of uniform normal surface tractions; the body is in equilibrium and may have cavities. *H. G. Hopkins.*

Lattanzi, Filippo. Applicazione della teoria dell'ellisse di elasticità trasversale allo studio di un'asta curva elasticamente vincolata agli estremi. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 178–186 (1951).

Cette note sert de complément à deux autres de même titre [mêmes Rend. **10**, 395–400; **11**, 45–52 (1951); ces Rev. **13**, 601, 796]. En supposant qu'on divise la barre élastique en parties suffisemment petites qu'on puisse les considérer comme rectilignes, on développe des formules sommatoires pour calculer les valeurs des coefficients dans les formules générales précédemment obtenues. *B. Levi* (Rosario).

Karunes, B. On the concentration of stress in the neighbourhood of a circular hole in a semi-infinite plate. Indian J. Phys. **25**, 599–606 (1951).

Using bipolar coordinates as defined by Jeffreys [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 265–293 (1920)] the author determines the Airy stress function for a semi-infinite plate containing a circular hole under (1) a uniform tension perpendicular to the straight edge and (2) a uniform shear in the plane of the plate. The hoop stress on the circular boundary is then found in the two cases as a slowly convergent infinite series, and its maximum value is determined for different ratios of the distance of the centre of the hole from the edge to the radius of the hole. *R. M. Morris* (Cardiff).

Sjöström, S. On the stresses at the edge of an eccentrically located circular hole in a strip under tension. Flygtekn. Försöksanstalt. Rep. no. **36**, 27 pp. (1950).

This paper deals with the stress in an infinite strip under tension along its length and containing an eccentrically located hole free from boundary stress. Starting from the Airy stress function for the stresses in an infinite plate under simple tension containing a circular hole, an auxiliary stress function is then determined by means of the Fourier integral theorem so that the stresses on the straight edges vanish. This auxiliary stress function introduces stresses on the edge of the hole which in turn are cancelled by another stress function which introduces stresses on the straight edges. Proceeding in this manner it is possible to make the remaining stresses on the edges arbitrarily small provided the procedure converges. The convergence is fairly rapid provided the diameter of the hole is less than the distance between the neighbouring edge and the centre of the hole. Numerical results are given for the maximum stress at the edge of the hole for different positions of the hole relative to the edges. *R. M. Morris* (Cardiff).

Higuchi, Masakazu. Calculation of the stresses of the orthotropic strip with a hole. Reports Res. Inst. Appl. Mech. Kyushu Univ. **1**, 33–45 (1952).

The stress of an orthotropic strip with a hole, circular or elliptical, is dealt with by the use of the stress function $\chi(x, y)=Z_1(z_1)+Z_2(z_2)+$comp. conj. where $z_r=x+ik_ry$, together with the stress formulae $\sigma_x=-\sum k_r^2Z_r''(z_r)$, $\sigma_y=\sum Z_r''(z_r)$, and $\tau_{xy}=-\sum ik_rZ_r''(z_r)$, where the summations range over four terms, two of them for $r=1, 2$ and the rest their conjugates. The function χ is first determined for a strip without a hole in the form of infinite integrals, the integrands of which can be determined in terms of the stress at infinity and the most general type of external force on the straight boundaries.

The orthotropic strip with an elliptic hole having one of its axes parallel to an edge is then treated, and as before the function χ is expressed in terms of infinite integrals whose integrands are then solutions of simultaneous Fredholm integral equations of the second kind involving Bessel's coefficients of odd order. These equations may be solved by the method of successive approximation and the case of an elliptic hole in a strip under tension parallel to its edges is treated in detail. Numerical results are given for strips of spruce and oak having circular holes and these are compared with the isotropic case and also the results for infinite plates of spruce and oak having circular holes. *R. M. Morris.*

Jung, H. Ein Beitrag zur Statik der Kreisplatten. Z. Angew. Math. Mech. **32**, 46–61 (1952).

The author employs the Hankel transformation to solve the differential equation associated with the deflection of a circular plate under rotationally-symmetric strains. It is remarked that this transformation applies equally well to a number of other problems in the theory of circular plates. *A. E. Heins* (Pittsburgh, Pa.).

Sengupta, H. M. On the bending of an elastic plate. II. Bull. Calcutta Math. Soc. **43**, 123–131 (1951).

In an earlier paper [same Bull. **41**, 163–172 (1949); these Rev. **11**, 287], the author solved the problem of a thin

clamped elliptic plate subjected to a normal concentrated load acting at any point on its surface. Making use of this solution and applying two symmetrically disposed loads, the author extends results to include (a) semi-elliptic plate clamped at elliptic edge and supported on major axis, (b) semi-elliptic plate clamped at elliptic edge and supported on minor axis, (c) quadrant of ellipse clamped at elliptic edge and supported on straight edges, each case with a single concentrated load at any point. Unfortunately, no numerical results are given. The author corrects errors and misprints in his earlier papers on elliptic plates.
H. D. Conway (Ithaca, N. Y.).

Pailloux, Henri. Statique et dynamique des membranes rigides. C. R. Acad. Sci. Paris **234**, 1430–1432 (1952).

The author sets up equations governing the statics of rigid membranes, using tensor analysis and functional derivatives. Thus he is able to put the results in the form of a Lagrangian problem. *C. Truesdell.*

Reissner, Eric. Stress strain relations in the theory of thin elastic shells. J. Math. Physics **31**, 109–119 (1952).

The stress-strain relations for thin shells based on "Love's first approximation" assume (a) that normals to the undeformed middle surface deform without change of length into the normals to the deformed middle surface, and that the stresses in the thickness direction are comparatively small, (b) that thickness/radii-of-curvature ratios are everywhere negligibly small compared with unity. By ignoring both of these assumptions, the author has previously obtained better approximations for homogeneous shells [Hildebrand, Reissner, and Thomas, Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1833** (1949); these Rev. **11**, 69] and non-homogeneous shells [Reissner, ibid., no. **1832** (1949); these Rev. **10**, 653]. Using a variational principle formulated for the purpose, the author repeats his investigation for homogeneous shells in a simpler manner. Improved stress-strain relations are then obtained by the use of a more general expression for the stresses in the thickness direction of the shell, and these include the relations of "Love's second approximation" as a special case. It is concluded that, although it is not sure that all correction terms have been obtained to Love's first approximation which are of the order of the thickness/radii-of-curvature ratios, nevertheless it is possible that all significant correction terms have been obtained. *H. D. Conway* (Ithaca, N. Y.).

Das Gupta, Sushil Chandra. Some simple problems of thick conical shells. Bull. Calcutta Math. Soc. **43**, 119–122 (1951).

Working directly from the equilibrium equations rather than using a stress function approach, the author solves problems of thick conical shells subjected to (a) an axial force, (b) torsion. It is interesting to note that problem (b) is given as an exercise in "Theory of elasticity", by Timoshenko and Goodier [2nd ed., McGraw-Hill, New York, 1951, p. 315; these Rev. **13**, 599]. *H. D. Conway.*

Muštari, H. M., and Surkin, R. G. On the nonlinear theory of the stability of elastic equilibrium of a thin spherical shell under the action of a uniformly distributed normal external pressure. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 573–586 (1950). (Russian)

The differences between the linear theory of Zoelly and others and experimental results have earlier been thoroughly investigated by several authors but so far no theory has given sufficiently accurate results. The authors revive this question and by keeping higher order terms and using a Ritz method succeed in giving a formula for the critical pressure which agrees better with experiments. The deformation during buckling is supposed to be a local depression which is known to happen in experiments. *F. Niordson.*

Radok, J. R. M. The theory of general instability of cylindrical shells. Coll. Aeronaut. Cranfield. Rep. no. 61, ii+16 pp. (1952).

A new approach is presented to the solution of the problem of the general instability of stiffened cylinders. The analysis is based on the usual differential equations of the small deflection theory for cylindrical shells. It is assumed that there is a linear distribution of infinite pressures and shears replacing the stringers and rings. The procedure for the general problem of discrete stringers and rings is outlined and stated to lead to a characteristic equation in the form of a determinant whose order is equal to three times the number of stringers and rings. The simplified problem of distributed stringers and discrete rings is solved in detail. There is no comparison of the results with those of previous theoretical and experimental investigation.
H. W. March (Madison, Wis.).

Baldacci, Riccardo F. Contributo alla dinamica della trave su appoggio elastico continuo. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 111–126 (1951).

The problem of the free vibrations of a finite, simply supported, elastic beam supported on an elastic foundation (whose resistance is directly proportional to surface displacement) is solved. The related problem of the forced vibrations of the beam produced by an applied transverse load, whose magnitude is simple harmonic in time and whose point of application moves at uniform speed, is solved; due attention is given to the special case of resonance. This problem is solved also for an infinite beam, the analysis being deduced as a limiting case of the previous analysis which itself is standard. In the reviewer's opinion the analysis could be made more concise through the use of operational methods, and at the same time greater generality in the applied load could be achieved with little additional work. *H. G. Hopkins* (Providence, R. I.).

Vlasov, V. Z. Some problems of the strength of materials, structural mechanics and the theory of elasticity. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 1267–1325 (1950). (Russian)

This paper sums up some earlier published papers by author. The first part contains fundamental results of the general theory of the strength, stability, and vibrations of thin-walled beams, space-systems of cylindrical and prismatic shells with open or closed sections. The author uses the assumption of an inextensible middle surface. The second part contains similar results for double-curved shells with the form determined by a polynomial of second degree. This theory is also applied to heliotechnical problems and to the construction of turbines. *F. Niordson.*

Raineri, Giuseppe. Solidi viscosi soggetti a distorsioni comunque variabili nel tempo. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 236–245 (1951).

The author claims to prove for what he calls a viscous solid "the independence of the final deformation from the manner in which the distorsion is varied in time." The reviewer is unable to understand the terms used or to follow the analysis. *C. Truesdell* (Bloomington, Ind.).

Savin, G. N., and Parasyuk, O. S. Some elastic-plastic problems with linear hardening. Ukrain. Mat. Žurnal 2, no. 1, 60–69, (1950). (Russian)

The authors consider a plastic material behaving according to the Hencky yield condition in a state of plane plastic strain. They seek solutions of the resulting equations which also satisfy the equilibrium conditions of the elastic plane strain problem. Solutions are found corresponding to the action of a concentrated force on a semi-infinite plane, a couple applied at the origin of an infinite plane, and an elastic-plastic problem previously solved by Galin [Akad. Nauk SSSR. Prikl. Mat. Meh. 10, 367–386 (1946); these Rev. 8, 241]. It is shown that other solutions exist.

H. I. Ansoff (Santa Monica, Calif.).

Drucker, D. C., and Prager, W. Soil mechanics and plastic analysis or limit design. Quart. Appl. Math. 10, 157–165 (1952).

This paper contains a consistent theory of stress and strain in the application of plasticity theory and limit design to problems of soil mechanics. The Mohr-Coulomb slip hypothesis for plane investigations in soil mechanics is generalized to a modified Mises yield criterion for a perfectly plastic material. The concept of a plastic potential is shown to imply the simultaneous occurrence of plastic deformation and dilatancy; this is a well-known physical feature of granular materials. This result implies in turn that the surfaces of discontinuity are planes and cylinders whose normal cross-section is a logarithmic spiral; in the extreme case when the yield stress in shear is independent of normal stress this cross-section is a circle, and this is the usual result for a Prandtl-Reuss material. As an example, the critical height of a vertical bank is treated by limit design techniques; the same problem is similarly treated for a soil that does not withstand tension, and lower critical heights then obtain. *H. G. Hopkins* (Providence, R. I.).

Malvern, L. E. Plastic wave propagation in a bar of material exhibiting a strain rate effect. Quart. Appl. Math. 8, 405–411 (1951).

The propagation of plastic deformation due to longitudinal impacts on a bar has been analyzed under the assumption that the relation between stress and strain remains the same as under static conditions. Comparisons with experiments have shown discrepancies due to the influence of the high rates of strain which occur in such impacts. Assuming a plastic stress strain relation of the form $\dot\sigma = E_0\dot\epsilon - g(\sigma, \epsilon)$ (E_0=Young's modulus), the author points out that the differential equations of motion can be integrated numerically essentially by the same method used for the static stress strain relation. The special case, $g=k(\sigma-\sigma_y)$, where σ_y is the yield stress is solved explicitly for an impact of constant velocity on a semi-infinite bar.

F. Bohnenblust (Pasadena, Calif.).

***Cotte, M. La mécanique des milieux piézoélectriques.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome IV. Études sur la mécanique des solides, études sur la mécanique générale, pp. 333–338. Publ. Sci. Tech. Ministère de l'Air, no. 261, Paris, 1952.

Eshelby, J. D. The force on an elastic singularity. Philos. Trans. Roy. Soc. London. Ser. A. 244, 87–112 (1951).

The author defines the force acting on an imperfection in an anisotropic elastic continuum as "the negative gradient of the total energy with respect to the position" of the imperfection in question, and develops a general method for calculating it. His memoir is comprehensive in character and presents a unified treatment of the subject, together with a wealth of new results. It is divided into ten sections, of which the first three are of an introductory nature.

In section 4, restricting his attention to continuous displacement fields and to continuous and single-valued stress fields, he defines a singularity in a finite body as follows: "Draw a closed surface Σ_0 in an infinite homogeneous elastic medium. We shall say that there is a singularity inside Σ_0 if the stresses in it could not be produced by body forces outside Σ_0. In section 5 he calculates the force exerted on a singularity by surface tractions. He shows that it does not depend on the shape of the body in which it is imbedded, but only on the surface tractions and on the displacement field of the singularity in an infinite medium. This work is applied in the succeeding section to derive the force exerted by surface tractions on edge and screw dislocations. The author obtains results which differ from those obtained by Koehler [Physical Rev. 60, 397–410 (1941)] by an incorrect procedure. The force on a general Somigliana dislocation is also exhibited.

In section 7, the author studies singularities with finite self-energy, and introduces two concepts. First, he defines the image force, which is a measure of the rate of decrease of elastic energy in a body with free surfaces, having a singularity S, when S is displaced. Second, he defines the force between singularities. He proves that the force F_l exerted on the singularities inside a closed surface Σ by (i) their image stresses, (ii) externally applied stresses, (iii) all the singularities outside Σ, is given by

$$(1) \qquad P_l = \int_\Sigma P_{jl}\, dS_j,$$

where dS_j is the surface element of Σ at a given point times the jth component of the outer normal to Σ at that point, and P_{jl} is the so-called Maxwell tensor of elasticity, analogous to the tensor of the same name in electrostatics.

In section 8, he considers anisotropic and inhomogeneous elastic bodies whose elastic constants $c_{ijlm} = c_{ijlm}(x_n - \xi_n)$, where the ξ_n's are parameters which could correspond, for example, to the position of a foreign particle in a body originally homogeneous. He defines the force on such inhomogeneities and shows that, in terms of this definition, it can be expressed by (1). Section 9 is devoted to the general case of a finite body having both singularities and inhomogeneities. Again, the total force F_l on these imperfections is given by (1). The author gives a critical analysis of Leibfried's [Z. Physik 127, 344–356 (1951); these Rev. 12, 304] calculation of the interaction energy between an impurity atom and an edge dislocation, which is in disagreement with his results, based on (1). Section 10 indicates the possible extension of the work in the preceding sections to the dynamic case. *A. W. Sáenz* (Bloomington, Ind.).

Nabarro, F. R. N. The interaction of screw dislocations and sound waves. Proc. Roy. Soc. London. Ser. A. 209, 278–290 (1951).

The paper falls naturally into two parts. Part I is devoted to the problem of scattering of plane transverse sound waves by screw dislocations, it being assumed that their wave-length is long compared to the interatomic spacing of the corresponding crystal lattice. The author solves this problem in an approximate manner by employing a dy-

namical extension of the Peierls dislocation model [R. Peierls, Proc. Phys. Soc. 52, 34–37 (1940)]. The scattering cross section obtained in this way is closely proportional to the wavelength. The momentum transfer cross-section is also calculated and is identical to the scattering cross-section above. This computation requires a knowledge of the forces between a variable elastic field and a moving dislocation; a topic which has been discussed recently by means of an electromagnetic analogy [see the preceding review].

In Part II the author analyzes Leibfried's theory [G. Leibfried, Z. Physik 127, 344–356 (1950); these Rev. 12, 304] of the interaction of dislocations with phonons in a critical fashion. He concludes on the basis of his results in Part I, that the latter's estimate of the resistance to the motion of dislocation arising from this interaction is much too large. It is hoped that some of the calculations in this paper will be carried out in a rigorous manner in future publications, because of their considerable interest to workers in dislocation theory as applied to solid state problems.
A. W. Sáenz.

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Suzuki, Tatsuro. The computation of the path of a ray and the correction of the aberrations of a lens system. I, II. Proc. Japan Acad. 27, 179–183, 184–187 (1951).

The tracing equations for meridian rays are derived in terms of the perpendicular from the center of curvature to the ray and the angle this perpendicular makes with the optical axis. With the help of these variables, differential formulas are set up for some of the image aberrations and an analytic procedure is suggested for undertaking the correction of the aberrations by small changes in the system data. *E. W. Marchand* (Rochester, N. Y.).

Theimer, O., Wassermann, G. D., and Wolf, E. On the foundation of the scalar diffraction theory of optical imaging. Proc. Roy. Soc. London. Ser. A. 212, 426–437 (1952).

An attempt is made to establish the validity of the scalar diffraction theory which, in contrast to the precise vector theory, assumes that the intensity may be taken as approximately equal to the square of the modulus of a single scalar wave function associated with the Hertzian vector. The present paper improves on the efforts of Picht (1925) in this direction, in that it takes into account the polarization and amplitude properties which an actual light source and the optical instrument impose upon the radiation field. Furthermore, the effect of the nonmonochromatic nature of natural light is considered. *E. W. Marchand* (Rochester, N. Y.).

Castoldi, Luigi. Sopra una proprietà del primo momento di una distribuzione spaziale limitata di dipoli o di multidipoli. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 35–42 (1950).

Agostinelli [Boll. Un. Mat. Ital. (2) 5, 1–5 (1943); these Rev. 7, 302] showed that a sphere which is magnetized parallel to a diameter has a total magnetic moment which is equal to the coefficient of the leading term in its expansion in spherical harmonics. In the present note the author generalizes the result to multidipoles of any order, and shows an equivalence between the moment of a distribution of dipoles and that of a single dipole. *J. W. Green.*

Denisov, N. G. Propagation of electromagnetic signals in an ionized gas. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 21, 1354–1363 (1951). (Russian)

This paper discusses the propagation of a finite wave train in a homogeneous ionized medium. The wave train is represented as a Fourier transform over all frequencies ω and then using the fact that the dielectric constant for waves of frequency ω is $\epsilon=1-a^2/\omega^2$, where a is a constant depending on the electron density, the author obtains an integral to represent the electromagnetic field produced by the signal. The integral is expressed in terms of Lommel functions of two variables and then, by means of certain approximations, simple expressions are obtained for the physically significant quantities. *B. Friedman.*

Skugarevskaya, O. A. On the initial stages of the process of establishing an electric current in a layer lying on an ideally conducting foundation. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1951, no. 6, 28–36 (1951). (Russian)

L'auteur étudie le stade initial de l'établissement du champ électrique dans une couche conductrice posée sur un substratum conducteur idéal. Dans le cas du substratum isolant le problème fut étudié par A. N. Tihonov [Izvestiya Akad. Nauk SSSR, Ser. Geograf. Geofiz. 14, 193–198, 199–222 (1950); ces Rev. 12, 29, 65]. Elle utilise sa méthode, en prenant comme point de départ les équations de Maxwell avec les conditions habituelles en ce qui concerne la continuité des composantes tangentielles des vecteurs **H** et **E** au passage d'une couche à l'autre. Pour le vecteur **H** on suppose que la circulation autour d'une ligne de courant correspond à l'intensité du courant. L'auteur obtient en fin de compte des formules maniables permettant de calculer le stade initial du processus. *V. A. Kostitzin* (Paris).

Skugarevskaya, O. A. On the final stages of the process of establishing an electric current in a layer lying on an ideally conducting foundation. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1951, no. 6, 37–49 (1951). (Russian)

Dans le mémoire analysé ci-dessus l'auteur a étudié le stade initial d'établissement du courant électrique dans une couche posée sur un substratum conducteur idéal. Ici en est étudié le stade final et sont déterminées pour t très grand les valeurs asymptotiques du champ s'établissant dans la couche. Elle considère deux schémas de disposition d'électrodes productrices et réceptrices du courant: parallèle et axiale. Les courbes asymptotiques obtenues sont confrontées avec celles données auparavant pour le stade initial, ceci pour les valeurs différentes de la profondeur de la couche.
V. A. Kostitzin (Paris).

Tihonov, A. N., and Skugarevskaya, O. A. On the establishment of an electric current in a nonhomogeneous stratified medium. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1951, no. 6, 50–55 (1951). (Russian)

Les auteurs donnent sans formules les résultats numériques et graphiques concernant l'établissement d'un champ électromagnétique dans un demi-espace stratifié hétérogène. Les électrodes productrices et réceptrices considérées comme infiniment petites se trouvent sur la surface. Quatre cas différents sont étudiés: 1) une couche d'épaisseur finie sur un substratum non-conducteur avec la disposition parallèle d'électrodes; 2) la même avec disposition axiale; 3) une

couche sur un substratum conducteur idéal, avec disposition parallèle; et 4) la même avec la disposition axiale d'électrodes.

V. A. Kostitzin (Paris).

Smythe, W. R. Flow over thick plate with circular hole. J. Appl. Phys. **23**, 447–452 (1952).

A uniform magnetic field is assumed parallel to the plane face of an infinite medium of infinite permeability (nonpermeable) into which circular hole is drilled normally. The magnetic scalar potential above the plane face is expressed as a series of oblate spheroidal harmonics, within the cylindrical hole as a series of Bessel functions. The determination of the first 11 coefficients of both series from the boundary conditions in the plane face of the medium bounding the cylindrical hole is carried through with various numerical checks. Computing the magnetic field along the axis of the cylindrical hole it is shown that its transverse value becomes insignificant for a depth equal to the diameter or more. The same solution is therefore used for a slab of thickness greater than twice the diameter of the hole and the same uniform magnetic field on both sides of the slab.

Approximating the wave equation by the Laplace equation, the author also applies the solution to the scatttering of a plane wave (with normal incidence) from an infinite flange of a circular pipe of infinite conductivity if the wavelength is large compared with the pipe diameter. Considering this flange as the wall of a resonant cavity in which a cylindrical hole is drilled normally (the previous pipe), an expression for the change in resonant frequency is derived. Finally, the same Laplace potential solution is interpreted as a problem in stationary current flow in a medium of finite resistivity to which is affixed normally a solid cylinder of the same resistivity.

E. Weber (Brooklyn, N. Y.).

***Mason, Max, and Weaver, Warren. The electromagnetic field.** Dover Publications, Inc., New York, N. Y., 1952. xiii+390 pp. Paperbound, $1.85; clothbound, $3.95.

Reprinted by photo-offset from the first edition [University of Chicago Press, 1929].

Hines, C. O. Electromagnetic energy density and flux. Canadian J. Physics **30**, 123–129 (1952).

This is a discussion of the relative merits of the Poynting theorem defining the flow density of electromagnetic field energy as $\mathbf{E}\times\mathbf{H}$ as compared with the Macdonald theorem [H. M. Macdonald, Electric waves, Cambridge Univ. Press, 1902] which substitutes $[\mathbf{E}\times\mathbf{H}+\frac{1}{2}\partial(\mathbf{A}\times\mathbf{H})/\partial t]$ and a further modification proposed by the author.

E. Weber (Brooklyn, N. Y.).

Jouvet, Bernard. La physique de l'Univers électromagnétique. C. R. Acad. Sci. Paris **234**, 1532–1534 (1952).

The author has previously [same C. R. **234**, 819–822 (1952); these Rev. **13**, 803] discussed a formulation of electromagnetic theory involving an eight-dimensional space, the topological product of space-time and the space of vector potentials defined over space-time. This paper is devoted to some remarks concerning the physical interpretation of this representation of electromagnetic theory. The gauge invariance of this theory is not considered.

A. H. Taub (Urbana, Ill.).

Quantum Mechanics

Gamba, Augusto. Vues sur les applications de la théorie des groupes à la physique quantique. Revue Sci. **90**, 11–24 (1952).

Halpern, Otto. A proposed re-interpretation of quantum mechanics. Physical Rev. (2) **87**, 389 (1952).

Bohm, David. Reply to a criticism of a causal re-interpretation of the quantum theory. Physical Rev. (2) **87**, 389–390 (1952).

Putnam, C. R. The quantum-mechanical equations of motion and commutation relations. Physical Rev. (2) **83**, 1047–1048 (1951).

It is shown that the commutation rules of quantum mechanics $[p, q]=h/i$ follow from the equation of motion if the Hamiltonian is of the form $H=\frac{1}{2}p^2+aq^n+b$ where n is an odd integer. This is a generalization of a result obtained for $n=1$ and $n=3$ by E. P. Wigner [same Rev. **77**, 711–712 (1950); these Rev. **11**, 706].

F. London.

Jauho, Pekka. On the harmonic oscillator with changed commutator. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **110**, 6 pp. (1952).

E. Wigner has pointed out that the Hamiltonian function and the equations of motion do not completely determine the commutators of canonically conjugated variables. In the present paper it is shown that in the case of the harmonic oscillator the commutator $[p, q]$ must have the form

$$[p, q]=\frac{h}{i}(1+\epsilon)$$

where ϵ is a diagonal matrix of the form

$$\epsilon_{nm}=(-1)^m \text{ const.}\times\delta_{nm}.$$

F. London (Durham, N. C.).

Bhabha, H. J. On a class of relativistic wave-equations of spin 3/2. Proc. Indian Acad. Sci., Sect. A. **34**, 335–354 (1951).

The author discusses a class of wave equations derivable from a Lagrangean

$$\psi^{+}D(\alpha^k p_k+\beta\chi)\psi \tag{1}$$

where $p_k=-i\partial/\partial x^k$, χ is an arbitrary constant, D, $D\alpha^k$, $D\beta$ are six hermitian matrices with certain transformation properties required by the invariance of (1) under all Lorentz transformations and ψ transforms under the representation $R(\frac{3}{2}, \frac{1}{2})+R(\frac{1}{2}, \frac{3}{2})$ of the full Lorentz group. He shows that the minimal equation satisfied by the α's is different from that satisfied by the matrices representing the infinitesimal generators of the group. He also shows that (1) there is only one equation in this class describing particles of finite mass in which the charge density is positive definite, namely, the equation equivalent to that proposed by Dirac, Fierz and Pauli for a particle of spin 3/2, (2) there is no equation of this class which describes a particle of spin 3/2 and zero rest mass, and (3) there is an equation in this class in which the particle has a state of finite rest mass and spin 3/2 and another state of zero rest mass and spin 1/2.

A. H. Taub (Urbana, Ill.).

Bhabha, H. J. An equation for a particle with two mass states and positive charge density. Philos. Mag. (7) **43**, 33–47 (1952).

Using the methods of the paper reviewed above, the author constructs a relativistic wave equation which describes a particle having two different rest masses. In one mass state the particle has a spin 3/2, in the other a spin 1/2. The charge density for every free wave solution of the equation is found to be positive definite, so that the equation can be quantized in the usual manner in accordance with the Pauli exclusion principle. It is shown that the equation is irreducible. *N. Rosen* (Chapel Hill, N. C.).

Kynch, G. J. The two-body scattering problem with non-central forces. I. Non-relativistic. Proc. Phys. Soc. Sect. A. **65**, 83–93 (1952).

A phase matrix $S(r)$ is introduced to treat two-body non-relativistic scattering problems with non-central forces. The asymptotic value of S, which is closely related to Heisenberg's scattering matrix, determines the scattering cross-section. $S(r)$ is shown to satisfy a first order Riccati equation. Approximate methods of solution are discussed. The formalism is applied to a comparison of neutron-proton and proton-proton scattering. The problem of bound states is also considered. *K. M. Case* (Los Alamos, N. M.).

Kynch, G. J. The two-body scattering problem with non-central forces. II. Relativistic. Proc. Phys. Soc. Sect. A. **65**, 94–101 (1952).

The method described in the paper reviewed above is applied to relativistic one- and two-particle scattering problems. It is shown that an S matrix can be introduced which satisfies a Riccati type equation similar to, and as simple in form, as those of the non-relativistic approximation. Methods of solution and bound state problems are discussed briefly. *K. M. Case* (Los Alamos, N. M.).

Freistadt, Hans. Sur l'hypothèse d'un intervalle fondamental et les théories de Darling et Born. C. R. Acad. Sci. Paris **235**, 23–25 (1952).

Raĭskiĭ, I. Quantum electrodynamics in a reciprocity formulation. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 194–199 (1952). (Russian)

A non-local spinor field theory is developed; its non-localizability is linked up with a constant which characterizes a non-vanishing rest mass. The electromagnetic field is considered localizable. Born's theory of reciprocity [Born and Green, Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 470–488 (1949); these Rev. **11**, 147] is used in introducing an interaction between the two fields, and in constructing an S-matrix which differs from the S-matrix of quantum electrodynamics by reciprocity factors of the type

$$\int d^4r \exp\{\pm(i/2)k_\nu r_\nu\}\rho(p, r)$$

(k_ν, p_ν energy-momentum four-vectors of photon and electron). The results obtained agree with those of ordinary quantum electrodynamics except when extreme relativistic energies are involved. Self-energy terms turn out to be finite, but, according to the author's opinion, they are still physically meaningless and should be removed by renormalization. *E. Gora* (Providence, R. I.).

Rayski, Jerzy. On field theories with non-localized interaction. Philos. Mag. (7) **42**, 1289–1297 (1951).

The author sets up a method of quantization of field equations which can be applied to a class of fields with a non-local interaction. He applies it to the extended source theory of Peierls and McManus [McManus, Proc. Roy. Soc. London. Ser. A. **195**, 323–336 (1948); these Rev. **10**, 664] and calculates the S-matrix for this case by the method of Yang and Feldman [Physical Rev. (2) **79**, 972–978 (1950); these Rev. **12**, 569]. He finds that the divergence difficulties still exist. He then considers another type of non-local interaction and finds that it is possible to get rid of the previous divergence difficulties. However, this involves giving up the localizability of charge and also the gauge invariance of the second kind. *N. Rosen.*

Belinfante, Frederik J., and Lomont, John S. Gauge-independent quantum electrodynamics. Physical Rev. (2) **84**, 541–546 (1951).

Quantum electrodynamics (positron theory) is formulated without the introduction of gauge-dependent potentials in a manner similar to that of Pauli [Handbuch der Physik, vol. XXIV/1, 2nd ed., Springer, Berlin, 1933, pp. 269–272] for the Dirac theory. The theory is developed in the interaction representation, the integrability of the Schrödinger equation is established, and the transformation to the Heisenberg representation discussed. The formulation is not obviously covariant. *H. C. Corben* (Genoa).

Belinfante, Frederik J. A variational principle for gauge-independent electrodynamics. Physical Rev. (2) **84**, 546–548 (1951).

Lagrangians are defined to describe classical and quantum electrodynamics in terms of gauge-independent transverse field strengths. In the quantum theory the derived commutation relations are shown to be those of the paper reviewed above. *H. C. Corben* (Genoa).

Belinfante, Frederik J. A new covariant auxiliary condition for quantum electrodynamics. Physical Rev. (2) **84**, 644–647 (1951).

The general solution of Maxwell's equations may be written as a sum of a retarded potential and a solution of the homogeneous equations, and it is postulated here that in the quantized theory the positive frequency part of the transverse part of this latter solution is an operator which gives zero when applied to any state vector, i.e. that for all states there are no photons from this part of the field present. Thus the expectation value of the field is always that of the retarded field. The covariance of this condition is established. *H. C. Corben* (Genoa).

Belinfante, Frederik J. The energy density tensor in gauge-independent quantum electrodynamics. Physical Rev. (2) **84**, 648–653 (1951).

Two different definitions of the energy momentum density tensor are given in the gauge-independent electrodynamics of the author [see the three preceding reviews] and the difficulty of rigorously establishing their equivalence is discussed. *H. C. Corben* (Genoa).

Iwata, Giiti. The unitary transformation and the quantization. Progress Theoret. Physics **7**, 39–44 (1952).

The author treats in a purely formal way various resemblences between the role of the group of contact trans-

formations in classical mechanics and that of the group of all unitary transformations in quantum field mechanics.
I. E. Segal (Chicago, Ill.).

Gora, E. Radiation reaction in relativistic motion of a particle in a wave field. Physical Rev. (2) 84, 1119–1123 (1951).

The classical equations of motion for a point charge proposed by P. A. M. Dirac [Proc. Roy. Soc. London. Ser. A. 167, 148–169 (1938)] are transformed by a change of variables. For the case of a circularly polarized monochromatic plane electromagnetic wave, they are solved approximately by an expansion which is valid if the effect of radiation reaction is small. To get rid of the "runaway" solutions [see C. J. Eliezer, Rev. Modern Physics 19, 147–184 (1947); these Rev. 9, 69], the author modifies the field so that the interaction sets in gradually, rather than abruptly at one moment of time. From the approximate solution, the transfer of energy and momentum between particle and field is calculated. The validity of the results obtained is examined, and the results are compared with those calculated according to quantum theory for a spinless particle. In the extreme relativistic domain agreement can be obtained if one assumes that in the classical case the particle absorbs the energy of one photon ($\hbar\omega$). Using this assumption to supplement the classical formulas, the author obtains an expression for the effective time of interaction between particle and field, which, he speculates, may be of some interest in meson theory.
N. Rosen (Chapel Hill, N. C.).

Daykin, P. N. An analysis of the self-energy problem for the electron in quantum electrodynamics. Canadian J. Physics 30, 70–78 (1952).

The well known treatment of the self-energy problem for the electron in quantum electrodynamics is based upon the assumption that only positive energy states are available to the virtual photons. Since up to now there is no conclusive evidence that this assumption is correct, the author investigates this problem assuming that the photon field is symmetrical in positive and negative energies. Feynman's simplified form of quantum electrodynamics is used. The distinction between one-electron theory and hole theory and between the different types of interaction is made by defining different contours for evaluating Feynman's S-matrix for the free electron self-energy. Pauli's result [Rev. Modern Physics 15, 175–207 (1943); these Rev. 5, 277] that this symmetric theory leads to a vanishing transverse part of the self-energy in one-electron theory is confirmed, but it is shown that a similar theorem does not hold in hole theory. A particular type of interaction proposed by Stueckelberg [Helvetica Phys. Acta 11, 225–244 (1938)] makes the self-energy in one-electron theory vanish altogether, but then other radiative corrections would vanish as well, and this would contradict experimental evidence like the Lamb-shift. The self-energy for the bound electron is evaluated in a similar manner, and decay probabilities are obtained from its imaginary part. Reasonable results arise only from the contour which corresponds to the hole theory of the positron and interaction with positive energy photons. *E. Gora.*

Eden, R. J. Threshold behaviour in quantum field theory. Proc. Roy. Soc. London. Ser. A. 210, 388–404 (1952).

The behaviour of the S-matrix of renormalized quantum field theory at thresholds for creation of particles of non-zero rest mass is investigated. The corresponding elements of the S-matrix are zero below such "creation thresholds" and non-zero above them. Since the S-matrix is unitary, one has to take into consideration additional "interference thresholds" for matrix elements which are in competition with the creation processes for the available probability; these matrix elements are non-zero on both sides of the boundary. This non-analytic behaviour raises the question whether it is possible to use methods of analytic continuation. The author shows that the elements of the S-matrix are analytic functions in regions where they are non-zero, and that they have branch points at interference thresholds. To obtain the path of continuation round the branch points it is assumed that the particles interact through their retarded fields. In order to avoid ambiguities an ϵ limiting process is used. Continuation round the singular points is prescribed by taking $\epsilon>0$ which corresponds to considering retarded fields. No difficulties of non-analytic behaviour arise for photon creation.
E. Gora (Providence, R. I.).

Le Couteur, K. J. Dirac's new electrodynamics. Nature 169, 146–147 (1952).

The basic ideas of the first form of Dirac's new classical electrodynamics are recalled [Proc. Roy. Soc. London 209, 291–296 (1951); these Rev. 13, 893]. The gauge equation is solved in the particular cases of (1) a Coulomb field, and (2) a free plane wave. Explicit expressions for the aether velocity are obtained. *A. J. Coleman.*

Born, Max. Dirac's new theory of the electron. Nature 169, 1105 (1952).

Yevick, George J. On the quantum theory for a finite-sized electron. I. Physical Rev. (2) 85, 911–917 (1952).

Different types of models for a finite-sized electron interacting with the electromagnetic field are first discussed from the classical point of view. Feynman's formalism [Rev. Modern Physics 20, 367–387 (1948); Physical Rev. 80, 440–457 (1950); these Rev. 10, 224; 12, 889] is then used to investigate the new problems encountered in quantizing such models. A theory is proposed wherein the electron and the photon propagate from one point in space-time to another in the usual way, and where the infinities are eliminated by a modification of the charge-current density only. The calculations are worked out in detail for the first-order perturbation. Emphasis is placed not so much on the elegance of the physical model but rather on the fact that Feynman's framework is "the most ideally suited at the present time to handle, in a relativistically covariant manner, the Lorentz type of finite-sized electron". *E. Gora.*

Gaus, Heinrich. Mesontheorie und Spin-Bahnkopplung im Kern. Z. Naturforschung 7a, 44–55 (1952).

In the theory of nuclear shell-structure it appears necessary to assume strong coupling between spin and orbital momentum of the nucleons. In a previous paper [same Z. 4a, 721–723 (1949)] the author has shown that such a coupling of the right order of magnitude has to be expected according to the neutral vector meson theory of nuclear forces. In an attempt to investigate the problem in a more general way, exchange effects and different types of meson fields are now taken into consideration. First, it is shown that one can still get a coupling of the correct order of magnitude if exchange effects are included, and if one chooses the charge-symmetrical vector theory. To represent exchange effects, the Hartree-Fock method is used. In a last section the author investigates the applicability of the scalar and pseudoscalar meson theories. In the scalar theory it is

possible to introduce a coupling between the meson field and the spin-density tensor of the nucleons which leads again to a spin-orbit interaction. In the pseudoscalar theory, on the contrary, such an interaction can be obtained only in higher approximations or in a non-linear version of the theory. *E. Gora* (Providence, R. I.).

Votruba, Václav. A generalized theory of the nucleon and meson. Rozpravy II. Třídy České Akad. **59**, no. 12, 20 pp. (1949). (Czech)

The wave equations for nucleons (Dirac's relativistic equation) and for mesons (Harish-Chandra's modification of the Duffin-Kemmer equations) are simultaneously generalized by replacing the rest masses by invariant and hermitian differential operators. These operators act upon an auxiliary scalar variable which appears in both the nucleon and the meson wave functions; they are defined in such a way that their characteristic values represent a positive, a discrete and monotonously increasing sequence of eigenvalues for the rest masses of nucleons and mesons in excited states which are suggested by this theory. A more detailed discussion of how these assumptions will modify the present formulation of nucleon-meson interaction and of the meson theory of nuclear forces is left for a subsequent paper.
E. Gora (Providence, R. I.).

Kwal, Bernard. Formulation rationnelle de la théorie des corpuscules de spin 1, en vue d'une théorie des mésons et des forces nucléaires. J. Phys. Radium (8) **12**, 868–872 (1951).

The author asserts that it is possible to set up theories of spin-one particles interacting with nucleons in a manner more completely analogous to the interaction of electrons and the electromagnetic field than the usual theory associated with Proca and Kemmer. He introduces a potential vector P_j and a potential tensor P_{jk} associated with the four-vector a_k and tensor h_{jk} as follows

$$a_k=\kappa P_k+\partial^j P_{jk}$$
$$h_{jk}=\kappa P_{jk}+\partial_j P_k-\partial_k P_j-\kappa^{-1}\partial^l[\partial_{[j}P_{kl]}].$$

Both P_k and P_{jk} admit gauge transformations which keep a_k and h_{jk} invariant. The equations satisfied by P_k and P_{jk} are

$$(\partial^j\partial_j-\kappa^2)P_k=g_1S_k$$
$$(\partial^j\partial_j-\kappa^2)P_{[jk]}=g_2S_{[jk]}$$

where S_k and S_{jk} are source densities. The interaction energy with a nucleon involve only the potentials, and the modified Dirac equation is

$$(\gamma^j\partial_j+\kappa_n)\psi=(i/\hbar c)[g_1\gamma^jP_j+(g_2/2)\gamma^{[jk]}P_{[jk]}]\psi.$$

For fixed nucleons the interaction will no longer involve the high inverse powers of the internucleon distance as it occurs in Kemmer theory. *H. Feshbach.*

Enatsu, Hiroshi. On the self-energies of nucleons. Progress Theoret. Physics **6**, 643–664 (1951).

The author makes use of the idea of Pais [Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. **19**, no. 1 (1947); these Rev. **8**, 554] and of S. Sakata and O. Hara [Progress Theoret. Physics **2**, 30–31 (1947)] of getting rid of divergences in the self-energies of particles by having two fields that contribute cancelling divergences. Using a covariant formalism, he finds the second-order self-energy of a nucleon due to the interaction with π mesons described by a symmetrical pseudo-scalar field with pseudo-vector coupling and with heavy mesons described by a symmetrical scalar field with vector coupling. According to the calculations, the divergences can be removed by choosing a suitable relation between the coupling constants and by assigning to the heavy meson a mass of 1474 electron masses. *N. Rosen.*

Enatsu, Hiroshi, and Pac, Pong Yul. On the mass difference of nucleons and the cohesive mesons. Progress Theoret. Physics **6**, 665–672 (1951).

Using the method of the paper reviewed above, but including the electromagnetic field, the authors find that, with a scalar field with scalar coupling and a pseudo-vector field with pseudo-vector coupling, it is possible to obtain the correct proton-neutron mass difference. However, the meson masses required are found to be quite small (110 and 13 electron masses), and the assumption of such particles appears to lead to disagreement with experiment.
N. Rosen (Chapel Hill, N. C.).

Katayama, Y. Transformation function in quantum electrodynamics. Progress Theoret. Physics **7**, 265–267 (1952).

Rideau, Guy. Au sujet des méthodes de Feynmann. C. R. Acad. Sci. Paris **234**, 1852–1855 (1952).

Marty, Claude. Contribution à l'étude covariante du champ nucléaire. Analyse des processus de diffusion nucléon-nucléon. Ann. Physique (12) **6**, 830–894 (1951).

This doctoral thesis consists of two parts. The first begins with a succinct and useful survey of experimental results on $n-p$ and $p-p$ scattering. This is followed by a resumé of phenomenological and non-relativistic meson theories of nuclear force which leads the author to the conviction that as far as nuclear forces are concerned we must abandon the concept of potential.

The second part begins with a brief derivation of Dyson's expansion of the S matrix and a development of the Feynman-Dyson rules for calculating its simpler terms. The emission and absorption factors corresponding to lines in the graph for scalar, pseudoscalar, vector and pseudovector mesons are conveniently tabulated. This theory is applied to the calculation of the cross-section for nucleon-nucleon scattering involving an arbitrary mixture of these meson fields (charged and neutral). The resulting formula requires fifteen lines to display! Particular sub-cases of this formula were given by Jean and Prentki [J. Phys. Radium **11**, 33–44 (1950)]. The calculation is to the first Born approximation with all higher order terms (in general, infinite) neglected. Except for the pseudoscalar case the formula obtained gives a cross-section of the proper order with small coupling constant, so the author argues that, in the "correct" theory whose existence is presupposed by current field theory, the higher order terms could be neglected. The pseudoscalar interaction requires too large a coupling constant for the present analysis to be applicable. Assuming that a pion of mass 276 is the main agent in nuclear force and that the above approximations are valid the observed $n-p$ scattering at 90 Mev. is found to be incompatible with a theory based on a scalar, vector, or pseudovector meson field or any combination of two such fields. *A. J. Coleman.*

Petiau, Gérard. Sur la représentation des systèmes d'équations d'ondes irréductibles de la théorie des corpuscules de spin quelconque. Application au calcul des sections efficaces de diffusion. C. R. Acad. Sci. Paris **234**, 1955–1957 (1952).

Rose, M. E., and Welton, T. A. The virial theorem for a Dirac particle. Physical Rev. (2) 86, 432–433 (1952).

Thermodynamics, Statistical Mechanics

✶Foà, Emanuele. Fondamenti di termodinamica. A cura di Arturo Giulianini. Nicola Zanichelli Editore, Bologna, 1951. xiii+255 pp. 3000 lire.

A traditional textbook presentation of the traditional material on thermostatics, rather more frank and more careful than average as far as the physics is concerned.

C. Truesdell (Bloomington, Ind.).

Caldirola, Piero. Le leggi fondamentali della meccanica statistica classica e quantistica. La meccanica statistica classica. I. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 11(80) (1947), 247–260 (1949).

ter Haar, D. The perfect Bose-Einstein gas in the theory of the quantum-mechanical grand canonical ensembles. Proc. Roy. Soc. London. Ser. A. 212, 552–558 (1952).

The theory of quantum-mechanical grand canonical ensembles is used to derive for the case of a perfect Bose-Einstein gas the average number of particles in the different energy levels, the fluctuations in these numbers and the equation of state. The Einstein condensation phenomenon is then discussed, and it is shown that in a p-v diagram (v being the specific volume) the isotherm consists of two analytically different parts in the limit where the number of particles in the system N goes to infinity. It is also shown that for finite N at the critical volume $\partial^n p/\partial v^n$ is of the order $N^{(n-2)/3}$ in accordance with a result obtained by Wergeland and Hove-Storhoug.

Author's summary.

Goldberger, M. L., and Adams, E. N., II. The configurational distribution function in quantum-statistical mechanics. J. Chem. Phys. 20, 240–248 (1952).

Operational techniques used by Schwinger [Physical Rev. 82, 664–679 (1951); these Rev. 12, 889] in electrodynamical problems are applied to statistical mechanics. Formulae are obtained which make feasible the continuation of the high temperature expansion of the distribution given by Wigner [ibid. 40, 749–759 (1932)] and Mayer and Band [same J. 15, 141–149 (1947)]. A low temperature expansion is attempted. The similarity of the equations obtained with those governing the slowing down of neutrons is exploited to give physical insight into the structure of the distribution function.

K. M. Case (Los Alamos, N. M.).

Fierz, M. Zur Theorie der Kondensation. Helvetica Phys. Acta 24, 357–366 (1951).

The author considers a gas consisting of atoms which attract each other. Restricting himself to the case of extremely small densities (far away from the critical point) he shows with the methods of statistical mechanics that this gas condenses at a correspondingly very low temperature. The point of condensation appears as a singularity of the equation of state at a well-defined pressure for which the density becomes indeterminate.

F. London (Durham, N. C.).

Kuhrt, Friedrich. Das Tröpfchenmodell realer Gase. Z. Physik 131, 185–204 (1952).

The properties of imperfect gases are treated by the method of the grand canonical ensemble. The equations obtained are formally identical with those of Mayer's theory [J. E. Mayer, J. Chem. Phys. 5, 67–73, 74–83 (1937)]. The cluster integral b_l represents the partition function of a droplet of l molecules. Asymptotic values for b_l in the limit of large l can be expressed by measurable properties of the liquid phase.

F. London (Durham, N. C.).

Kuhrt, Friedrich. Das Tröpfchenmodell übersättigter realer Gase. Z. Physik 131, 205–214 (1952).

Results of the preceding paper concerning the number of droplets in a vapor are applied to the metastable oversaturated state. The reviewer would question the legitimacy of this application since the grand canonical ensemble can represent only equilibrium states.

F. London.

McLellan, A. G. A new method of solving the Born-Green equation for the radial distribution function. Proc. Roy. Soc. London. Ser. A. 210, 509–517 (1952).

A solution of the Born-Green equation for the radial distribution function is given in form of a series in ascending powers of the density. The method is applied to a fluid of rigid spherical molecules and the results are compared with those which Kirkwood, Maun and Alder [J. Chem. Phys. 18, 1040–1047 (1950); these Rev. 13, 196] calculated by means of numerical integration.

F. London.

Ono, Syû. Integral equations between distribution functions of molecules. Progress Theoret. Physics 5, 822–832 (1950).

Ono, Syû. Statistical thermodynamics of solutions of electrolytes and non-electrolytes. Progress Theoret. Physics 6, 447–457 (1951).

Ono, Syû. Statistical mechanics of adsorption from multicomponent systems. I. J. Phys. Soc. Japan 6, 10–15 (1951).

These three articles contain discussions and applications of the theory of configurational distribution functions of classical systems of interacting particles. The connections between the approaches of Kirkwood [J. Chem. Phys. 3, 300–313 (1935)], Mayer [ibid. 15, 187–201 (1947)], and Born and Green [A general kinetic theory of liquids, Cambridge Univ. Press, 1949; these Rev. 12, 230] are discussed in the first paper.

A theory of solutions is presented in the second paper in terms of the integral-differential equations for the distribution functions of positions of particles. It is shown that the Debye-Hückel theory of electrolytes follows from the general equations after an application of the Kirkwood superposition principle and the linearization of the resulting equations for the two particle distribution function.

The third paper contains the application of the general theory to the absorption of gases on solid surfaces. Equations are derived for the molecular distribution function of gas molecules near the surface and for the absorption isotherms.

E. W. Montroll (College Park, Md.).

BIBLIOGRAPHICAL NOTES

Annales Universitatis Saraviensis.

Vol. 1, no. 1, of this journal appeared in 1952. It is published by the Université de la Sarre (Universität des Saarlandes), Saarbrücken.

Bulletin of Mathematical Statistics.

Beginning with vol. 4, nos. 1–2, dated December 1950, this Bulletin is being published in English. It is published by the Research Association of Statistical Sciences, Kyushu University, Fukuoka, Japan.

Časopis pro Pěstování Matematiky.

This is a continuation of the former Časopis pro Pěstování Matematiky a Fysiky. The first number under the new title carries the volume number 76, continuing the numbering of the old Časopis. The content is different from that of the Czechoslovak Mathematical Journal and is devoted chiefly to expository lectures, book reviews and news of interest to Czechoslovakian mathematicians.

Czechoslovak Mathematical Journal.

Vol. 1, no. 1, of this journal is dated September, 1951. This journal is also successor to the Časopis pro Pěstování Matematiky a Fysiky which terminated with vol. 75, no. 4. In addition to this version of the journal which will be written in English, French, or German, there will be a version in Russian, Čehoslovackiĭ Matematičeskiĭ Žurnal.

Mathematical Journal of Okayama University.

Vol. 1, nos. 1–2, of this journal is dated March, 1952. It is published by the Department of Mathematics, Faculty of Science, Okayama University, Okayama, Japan.

Natural Science Report of the Ochanomizu University.

Vol. 1 of this journal appeared in 1951. It is published by Ochanomizu University, Tokyo, Japan.

Proceedings of the Glasgow Mathematical Association.

Vol. 1, part 1, of this journal is dated January, 1952.

Rendiconti del Circolo Matematico di Palermo.

Vol. 1, no. 1, of the 2d series of this journal appeared in 1952. Its address is Via Archirati 34, Palermo, Italy.

Ricerche di Matematica.

Vol. 1, no. 1, of this journal appeared in 1952. It is published by the Istituto di Matematica dell'Università, Via Mezzocannone 8, Naples.

Scientific Papers of the College of General Education. University of Tokyo.

Vol. 1, no. 1, of this publication is dated October, 1951. The address is Komaba, Meguro-ku, Tokyo, Japan.

The Michigan Mathematical Journal.

Vol. 1, no. 1, of this journal is dated January, 1952. It is published by the University of Michigan Press, Ann Arbor, Mich.

The Saugar University Journal.

Vol. 1, no. 1, of this journal is dated 1951–52. It is published by the University of Saugar, Sagar, India.

Trudy Moskovskogo Matematičeskogo Obščestva.

The first volume of these Trudy appeared in 1952. It will print the papers of mathematical interest presented at the meetings of the Moscow Mathematical Society. It is published by the Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury.

DECEMBER ISSUE IS AN INDEX WHICH HAS BEEN PHOTOGRAPHED AT THE BEGINNING OF THE VOLUME(S).